Higher

Mathematics

Sequences

Contents

CfE Edition

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Sequences

1 Introduction to Sequences A

A **sequence** is an ordered list of objects (usually numbers).

Usually we are interested in sequences which follow a particular pattern. For example, $1, 2, 3, 4, 5, 6, \ldots$ is a sequence of numbers – the "…" just indicates that the list keeps going forever.

Writing a sequence in this way assumes that you can tell what pattern the numbers are following but this is not always clear, e.g.

$$28,\ 22,\ 19,\ 17\tfrac{1}{2},\ \ldots.$$

For this reason, we prefer to have a formula or rule which explicitly defines the terms of the sequence.

It is common to use subscript numbers to label the terms, e.g.

$$u_1,\ u_2,\ u_3,\ u_4,\ \ldots$$

so that we can use u_n to represent the nth term.

We can then define sequences with a formula for the nth term. For example:

Formula	List of terms
$u_n = n$	$1,\ 2,\ 3,\ 4,\ \ldots$
$u_n = 2n$	$2,\ 4,\ 6,\ 8,\ \ldots$
$u_n = \frac{1}{2}n(n+1)$	$1,\ 3,\ 6,\ 10,\ \ldots$
$u_n = \cos\left(\frac{n\pi}{2}\right)$	$0,\ -1,\ 0,\ 1,\ \ldots$

Notice that if we have a formula for u_n, it is possible to work out *any* term in the sequence. For example, you could easily find u_{1000} for any of the sequences above without having to list all the previous terms.

Recurrence Relations

Another way to define a sequence is with a **recurrence relation**. This is a rule which defines each term of a sequence using previous terms.

For example:

$$u_{n+1} = u_n + 2, \; u_0 = 4$$

says "the first term (u_0) is 4, and each other term is 2 more than the previous one", giving the sequence $4, 6, 8, 10, 12, 14, \ldots$.

Notice that with a recurrence relation, we need to work out all earlier terms in the sequence before we can find a particular term. It would take a long time to find u_{1000}.

Another example is interest on a bank account. If we deposit £100 and get 4% interest per year, the balance at the end of each year will be 104% of what it was at the start of the year.

$$\begin{aligned} u_0 &= 100 \\ u_1 &= 104\% \text{ of } 100 = 1{\cdot}04 \times 100 = 104 \\ u_2 &= 104\% \text{ of } 104 = 1{\cdot}04 \times 104 = 108{\cdot}16 \\ &\vdots \end{aligned}$$

The complete sequence is given by the recurrence relation

$$u_{n+1} = 1{\cdot}04 u_n \text{ with } u_0 = 100,$$

where u_n is the amount in the bank account after n years.

EXAMPLE

The value of an endowment policy increases at the rate of 5% per annum. The initial value is £7000.

(a) Write down a recurrence relation for the policy's value after n years.

(b) Calculate the value of the policy after 4 years.

2 Linear Recurrence Relations A

In Higher, we will deal with recurrence relations of the form

$$u_{n+1} = au_n + b$$

where a and b are any real numbers and u_0 is specified. These are called **linear recurrence relations** of order one.

Note

To properly define a sequence using a recurrence relation, we must specify the initial value u_0.

EXAMPLES

1. A patient is injected with 156 ml of a drug. Every 8 hours, 22% of the drug passes out of his bloodstream. To compensate, a further 25 ml dose is given every 8 hours.

   (a) Find a recurrence relation for the amount of drug in his bloodstream.

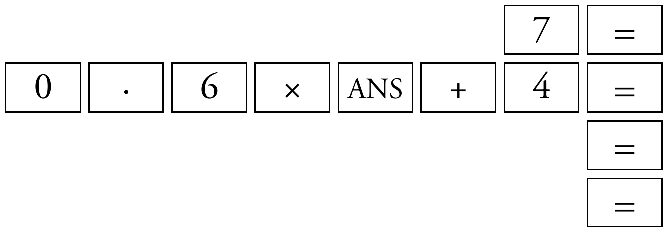
   (b) Calculate the amount of drug remaining after 24 hours.

2. A sequence is defined by the recurrence relation $u_{n+1} = 0{\cdot}6u_n + 4$ with $u_0 = 7$.

   Calculate the value of u_3 and the smallest value of n for which $u_n > 9{\cdot}7$.

Using a Calculator

Using the ANS button on the calculator, we can carry out the above calculation more efficiently.

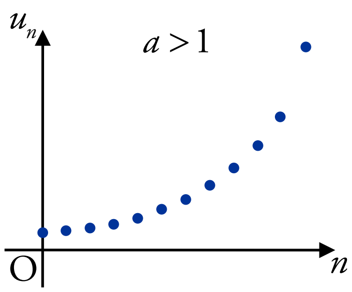


3 Divergence and Convergence A

If we plot the graphs of some of the sequences that we have been dealing with, then some similarities will occur.

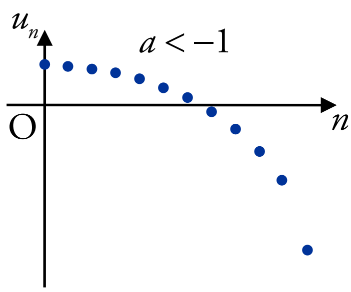
Divergence

Sequences defined by recurrence relations in the form $u_{n+1} = au_n + b$ where $a < -1$ or $a > 1$, will have a graph like this:



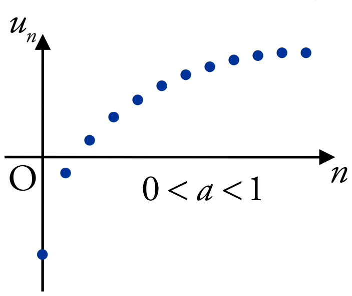
Sequences like this will continue to increase or decrease forever.

They are said to **diverge**.



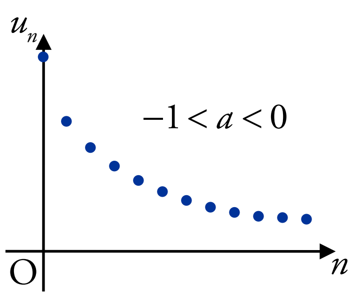
Convergence

Sequences defined by recurrence relations in the form $u_{n+1} = au_n + b$ where $-1 < a < 1$, will have a graph like this:



Sequences like this "tend to a limit".

They are said to **converge**.



4 The Limit of a Sequence A

We saw that sequences defined by $u_{n+1} = au_n + b$ with $-1 < a < 1$ "tend to a limit". In fact, it is possible to work out this limit just from knowing a and b.

The sequence defined by $u_{n+1} = au_n + b$ with $-1 < a < 1$ tends to a limit l as $n \to \infty$ (i.e. as n gets larger and larger) given by

$$l = \frac{b}{1-a}.$$

You will need to know this formula, as it is not given in the exam.

EXAMPLES

1. The deer population in a forest is estimated to drop by 7·3% each year. Each year, 20 deer are introduced to the forest. The initial deer population is 200.

   (a) How many deer will there be in the forest after 3 years?

   (b) What is the long term effect on the population?

2. A sequence is defined by the recurrence relation $u_{n+1} = ku_n + 2k$ and the first term is u_0.
   Given that the limit of the sequence is 27, find the value of k.

5 Finding a Recurrence Relation for a Sequence A

If we know that a sequence is defined by a linear recurrence relation of the form $u_{n+1} = au_n + b$, and we know three consecutive terms of the sequence, then we can find the values of a and b.

This can be done easily by forming two equations and solving them simultaneously.

EXAMPLE

A sequence is defined by $u_{n+1} = au_n + b$ with $u_1 = 4$, $u_2 = 3{\cdot}6$ and $u_3 = 2{\cdot}04$.

Find the values of a and b.