Unit 3

Vectors

- Vectors are named using a directed line segment, eg $\overrightarrow{AB}$, or a bold letter, eg $\boldsymbol{u}$, written by hand as $\underline{u}$
- A component vector is in the form $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ – also known as a column vector
- The magnitude of a vector, denoted $|\overrightarrow{AB}|$ or $|\boldsymbol{u}|$, is $\sqrt{a^2+b^2+c^2}$
- Multiplication by a scalar is $k\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}$.

If $\boldsymbol{u} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}$ then $2\boldsymbol{u} = \boldsymbol{v}$

this means that $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel, but $\boldsymbol{v}$ is twice as long as $\boldsymbol{u}$

- The zero vector is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
- $\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

eg $3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

- $\boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix}$
- $\boldsymbol{u} - \boldsymbol{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a-d \\ b-e \\ c-f \end{pmatrix}$
- $\overrightarrow{BA}$ is the negative of $\overrightarrow{AB}$.

eg $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow \overrightarrow{BA} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$

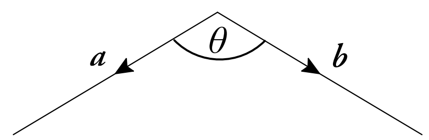
- This is the same line segment, but it points in the opposite direction

The Scalar Product

$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

- $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ (given in the exam)

Remember: the vectors must point away from the vertex, eg



- $\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$ (given in the exam)
- $\cos\theta = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$
- $\cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{|\boldsymbol{a}||\boldsymbol{b}|}$
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ since $\cos 90° = 0$
- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$ and $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$

Position Vectors

- $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are the position vectors of A and B.

Collinearity

- If $\overrightarrow{AB} = k\overrightarrow{BC}$, where k is a scalar, then $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$. If B is a common point then A, B and C are collinear.
- To find the coordinates of a point B which divides $\overrightarrow{AC}$ in the ratio 2:3, use

$$\frac{\overrightarrow{AB}}{\overrightarrow{BC}} = \frac{2}{3}$$

Further Calculus

- $y = \sin ax \Rightarrow \dfrac{dy}{dx} = a\cos ax$ $\quad$ $y = \cos ax \Rightarrow \dfrac{dy}{dx} = -a\sin ax$
- $\int \sin ax \; dx = -\frac{1}{a}\cos ax + C$ $\quad$ $\int \cos ax \; dx = \frac{1}{a}\sin ax + C$

The above are given in the exam

- **Chain Rule**

$y = (ax+b)^n \Rightarrow \dfrac{dy}{dx} = an(ax+b)^{n-1}$

"power multiplies to the front, bracket stays the same, lower the power by 1 and multiply by the derivative of the bracket"

Example

$y = (x^9 + 8x)^3$

$\dfrac{dy}{dx} = 3(x^9+8x)^2 \times (9x^8+8)$

$= 3(9x^8+8)(x^9+8x)^2$

- $\int (ax+b)^n \; dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$

"Raise the power by 1 and divide by the new power multiplied by the derivative of the bracket, plus C"

The "derivative of the bracket" should only ever be an integer, eg 3 not $4x^2$ etc.

Example

$\int (2x+3)^4 \; dx = \dfrac{(2x+3)^5}{5 \times 2} + C = \dfrac{(2x+3)^5}{10} + C$

Logarithms and Exponentials

- A function in the form $y = a^x$ is called an exponential function
- e^x is called the exponential function to the base e.
- $y = a^x \Rightarrow \log_a y = x$ – "the base number stays the same, and the other two terms flip over"

Laws of Logs

- $\log_a x + \log_a y = \log_a xy$ (Squash)
- $\log_a x - \log_a y = \log_a \frac{x}{y}$ (Split)
- $\log_a x^n = n\log_a x$ (Fly)
- $\log_a a = 1$ eg $\log_8 8 = 1$
- $\log_e x$ is the same as $\ln x$ and is called the natural logarithm
- 'log' on a calculator stands for $\log_{10}$ and 'ln' stands for $\log_e$
- To solve an equation where the unknown is a power, you must take logs of both sides and use the 'fly' rule. It does not matter if you use $\log_{10}$ or ln but if the equation involves an e then ln could be easier.

Examples

$5^x = 11$

$\log_{10} 5^x = \log_{10} 11$

$x\log_{10} 5 = \log_{10} 11$

$x = \dfrac{\log_{10} 11}{\log_{10} 5}$

$= 1.49$ (to 2 d.p.)

$e^x = 14$

$\ln e^x = \ln 14$

$x\ln e = \ln 14$

$x = \ln 14$ (since $\ln e = \log_e e = 1$)

$= 2.64$ (to 2 d.p.)

Experimental Data

Must use the laws of logs to write in the form $y = mx + c$.
See the notes for more detail.

The Wave Function

- $a\cos x + b\sin x$ can be expressed in the form $k\cos(x-\alpha)$ where

$$k = \sqrt{a^2+b^2} \text{ and } \tan\alpha = \frac{k\sin\alpha}{k\cos\alpha}$$

- $a\cos x + b\sin x$ can also be expressed as $k\cos(x+\alpha)$, $k\sin(x+\alpha)$ or $k\sin(x-\alpha)$ but if you have a choice, $k\cos(x-\alpha)$ is always more straightforward.
- $a\cos 2x + b\sin 2x$ can be written in any of the forms; ie $k\cos(2x-\alpha)$ etc
- If asked to solve an equation which contains both a $\sin x$ and a $\cos x$, must use either:
  Double angle substitution (see U2OC3), or
  The wave function

The solution must involve a CAST diagram.

Example $\quad$ Express $3\cos x + 4\sin x$ in the form $k\cos(x-\alpha)$

$k\cos(x-\alpha) = k\cos x\cos\alpha + k\sin x\sin\alpha$

$= k\cos\alpha\cos x + k\sin\alpha\sin x$

$k\cos\alpha = 3$

$k\sin\alpha = 4$

✓ S	A ✓✓
T	C ✓

Hence α is in the first quadrant

$k = \sqrt{3^2+4^2}$

$= \sqrt{25}$

$= 5$

$\tan\alpha = \dfrac{k\sin\alpha}{k\cos\alpha}$

$= \frac{4}{3}$

$\alpha = 53.1°$

So $3\cos x + 4\sin x = 5\cos(x - 53.1)°$