Higher

Mathematics

UNIT 1 OUTCOME 4

Sequences

Contents

HSN21400

OUTCOME 4

Sequences

1 Introduction to Sequences

A sequence is a list of numbers which follow a pattern. To save writing out a list each time, we can define a sequence using a rule or formula.

Recurrence Relations

Recurrence relations are one way of describing sequences. These give the nth term of a sequence in terms of the $(n-1)$th term.

Consider the example where we are adding on 4% interest annually to £1000 for a 6 year period.

When adding on 4% to the initial value, we could do the following:

$$\text{Year 1 Interest} = 4\% \text{ of } 1000 = 0{\cdot}04 \times 1000 = 40$$

$$\text{Total} = 1000 + 40 = 1040$$

Calculating £1040 by this method is the same as calculating 104% of the initial value.

In recurrence relations, we use u_0 to stand for the initial value. Therefore the first term of the sequence is u_1.

$$u_0 = 1000$$

$$u_1 = 104\% \text{ of } 1000 = 1{\cdot}04 \times 1000 = 1040$$

The recurrence relation for this sequence can be written as

$$u_{n+1} = 1{\cdot}04u_n \qquad \text{with } u_0 = 1000$$

where u_n is the value after n years, and u_{n+1} is the value after $n+1$ years.

EXAMPLE

The value of an endowment policy increases at the rate of 5% per annum. The initial value is £7000.

(a) Write down a recurrence relation for the policy's value after n years.

(b) Calculate the value of the policy after 4 years.

(a) Let u_n be the value of the policy after n years.

So $u_{n+1} = 1{\cdot}05u_n$ with $u_0 = 7000$

(b) $u_0 = 7000$

$u_1 = 1{\cdot}05 \times 7000 = 7350$

$u_2 = 1{\cdot}05 \times 7350 = 7717{\cdot}5$

$u_3 = 1{\cdot}05 \times 7717{\cdot}5 = 8103{\cdot}375$

$u_4 = 1{\cdot}05 \times 8103{\cdot}375 = 8508{\cdot}54375$

After 4 years, the policy is worth £8508·54

2 Linear Recurrence Relations

Linear recurrence relations are of the form

$$u_{n+1} = au_n + b$$

where $a \neq 0,\ b \in \mathbb{R}$.

Note

To properly define a sequence using a recurrence relation, we must specify the initial value u_0.

EXAMPLES

1. A patient is injected with 156 ml of a drug. Every 8 hours, 22% of the drug passes out of his bloodstream. To compensate, a further 25 ml dose is given every 8 hours.

   (a) Find a recurrence relation for the amount of drug in his bloodstream.

   (b) Calculate the amount of drug remaining after 24 hours.

   (a) Let u_n be the amount of drug in his bloodstream after $8n$ hours.

   $u_{n+1} = 0{\cdot}78u_n + 25$ with $u_0 = 156$

   (b) $u_0 = 156$

   $u_1 = 0{\cdot}78 \times 156 + 25 = 146{\cdot}68$

   $u_2 = 0{\cdot}78 \times 146{\cdot}68 + 25 = 139{\cdot}4104$

   $u_3 = 0{\cdot}78 \times 139{\cdot}4104 + 25 = 133{\cdot}7401$

   After 24 hours, he will have 133.74 ml of drug in his bloodstream.

2. A sequence is defined by the recurrence relation $u_{n+1} = 0{\cdot}6u_n + 4$ with $u_0 = 7$.

   Calculate the value of u_3 and the smallest value of n for which $u_n > 9{\cdot}7$.

$u_0 = 7$

$u_1 = 0{\cdot}6 \times 7 + 4 = 8{\cdot}2$

$u_2 = 0{\cdot}6 \times 8{\cdot}2 + 4 = 8{\cdot}92$

$u_3 = 0{\cdot}6 \times 8{\cdot}92 + 4 = 9{\cdot}352$

The value of u_3 is $9{\cdot}352$

$u_4 = 9{\cdot}6112$

$u_5 = 9{\cdot}76672$

The smallest value of n for which $u_n > 9{\cdot}7$ is 5

Using a Calculator

Using the ANS button on the calculator, we can carry out the above calculation more efficiently.

						7	=
0	·	6	×	ANS	+	4	=
							=
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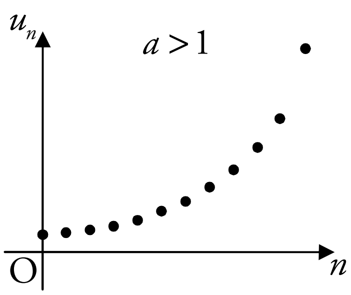
3 Divergence and Convergence

If we plot the graphs of some of the sequences that we have been dealing with, then some similarities will occur.

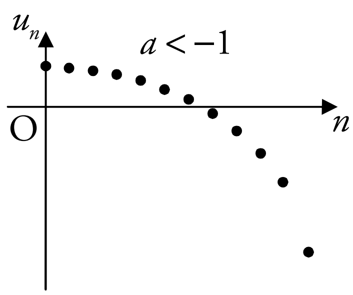
Divergence

Sequences defined by recurrence relations in the form $u_{n+1} = au_n + b$ where $a < -1$ or $a > 1$, will have a graph like this:



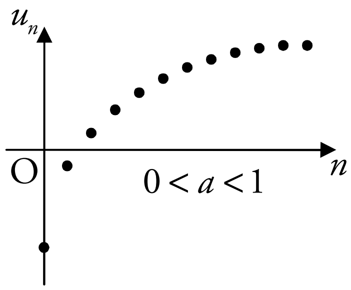
Sequences like this will continue to increase or decrease forever.

They are said to diverge.



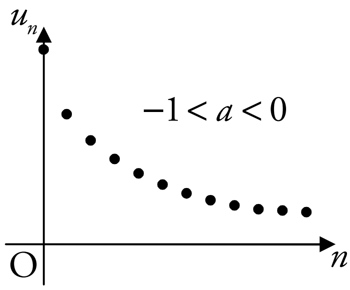
Convergence

Sequences defined by recurrence relations in the form $u_{n+1} = au_n + b$ where $-1 < a < 1$, will have a graph like this:



Sequences like this "tend to a limit".

They are said to converge.



4 The Limit of a Sequence

For convergent sequences defined by $u_{n+1} = au_n + b$ with $-1 < a < 1$, u_n tends to a limit l as $n \to \infty$ (i.e. as n gets larger and larger) and:

$$l = \frac{b}{1-a} \quad \text{where } -1 < a < 1$$

You will need to know this formula, as it is not given in the exam.

EXAMPLE

The deer population in a forest is estimated to drop by 7·3% each year. Each year, 20 deer are introduced to the forest. The initial deer population is 200.

(a) How many deer will there be in the forest after 3 years?

(b) What is the long term effect on the population?

(a) $u_{n+1} = 0{\cdot}927u_n + 20$

$$u_0 = 200$$
$$u_1 = 0{\cdot}927 \times 200 + 20 = 205{\cdot}4$$
$$u_2 = 0{\cdot}927 \times 205{\cdot}4 + 20 = 210{\cdot}4058$$
$$u_3 = 0{\cdot}927 \times 210{\cdot}4058 + 20 = 215{\cdot}0461$$

Therefore there are 215 deer living in the forest after 3 years.

(b) A limit exists, since $-1 < 0{\cdot}927 < 1$

$$l = \frac{b}{1-a} \quad \text{where } a = 0{\cdot}927 \text{ and } b = 20$$
$$= \frac{20}{1-0{\cdot}927}$$
$$= 273{\cdot}97 \text{ (to 2 d.p.)}$$

Therefore the number of deer in the forest will settle around 273.

Note
Whenever you calculate a limit, you must state that "A limit exists since $-1 < a < 1$"

5 Solving Recurrence Relations to find *a* and *b*

If we know that a sequence is defined by a linear recurrence relation of the form $u_{n+1} = au_n + b$, and we know several terms of the sequence, then we can find the values of a and b.

This can be done easily by solving the equations simultaneously.

EXAMPLE

A sequence is defined by $u_{n+1} = au_n + b$ with $u_1 = 4$, $u_2 = 3{\cdot}6$ and $u_3 = 2{\cdot}04$.

Find the values of a and b.

Find equations for two values of n:

$$u_2 = au_1 + b$$
$$u_3 = au_2 + b$$

$$3{\cdot}6 = 4a + b$$
$$2{\cdot}04 = 3{\cdot}6a + b$$

$$4a + b = 3{\cdot}6$$
$$-3{\cdot}6a - b = -2{\cdot}04$$

Solve for a by eliminating b:

$$\begin{aligned} 4a + b &= 3{\cdot}6 \\ -3{\cdot}6a - b &= -2{\cdot}04 \\ \hline 0{\cdot}4a &= 1{\cdot}56 \end{aligned}$$

$$a = \frac{1{\cdot}56}{0{\cdot}4}$$
$$a = 3{\cdot}9$$

Substitute $a = 3{\cdot}9$ into equation:

$$4(3{\cdot}9) + b = 3{\cdot}6$$
$$15{\cdot}6 + b = 3{\cdot}6$$
$$b = 3{\cdot}6 - 15{\cdot}6$$
$$b = -12$$

So $a = 3{\cdot}9$ and $b = -12$.