Introducing… Proof

Contents

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1 Introduction

So far in school, mathematics has been mainly about carrying out calculations and solving numerical problems. Sometimes we will have had to use theorems to solve a problem – for example Pythagoras's Theorem.

But how do we know that a rule or theorem is correct? How can we be sure that it will work?

What is proof?

We are quite familiar with the concept of "proof" in everyday life. A **proof** is a convincing argument that statement is true. But when is an argument convincing *enough* to be convincing?

In a court of law, the guilt or innocence of a defendant is proven. Lawyers present evidence for each side and a jury decides which argument is convincing. If a jury is convinced that a defendant is guilty, then it has been proven that he is guilty – regardless of whether or not he actually committed the crime.

In mathematics, there can be no uncertainty. The proof of a mathematical statement must be a correct logical argument. We can only properly prove things that really *are* true!

2 Statements, Definitions and Euler Diagrams

When we do something for the first time, we usually start by learning some terminology. Constructing a proof is no different.

Statements

As far as we are concerned, a **statement** is a sentence or clause (part of a sentence) which must be either true or false (but not both at once).

For example, a statement in English is:

I will go to school today.

This statement must be either true or false (but not both at once). Note that depending on who says this statement it may always be false!

Statements may also be mathematical in nature:

In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

This is Pythagoras's Theorem. This statement is always true – we will prove it later!

To save writing out an entire statement each time, we usually label it. We will label statements using uppercase letters. For example we could write:

A I will go to school today.

After labelling a statement, we can then say "A" and this means the same thing as saying "I will go to school today". Labels just make referring to statements easier.

Statements have a **truth value** which is either *true* or *false* but not both.

For example, the truth value of Pythagoras's Theorem (above) is *true* and the statement "$1 = 0$" is *false*.

We will only consider statements which can be assigned a truth value.

Definitions

To begin with we will look at some simple mathematical statements which we can prove directly from a definition.

A **definition** is a statement of what a term or phrase means.

All a definition does is allows us to refer to a concept more concisely.

For example, rather than saying "determine whether the statement A is true or false", we can say "determine the *truth value* of A".

(In this document, all definitions are formatted as above.)

Our first proof

Here we will prove some statements about integers. Just in case you are unfamiliar with this term, here is a definition:

An **integer** is any one of the numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.

The symbol "$\ldots$" is called an ellipsis. In mathematics, it means that the pattern is continued.

We will also be concerned with integers being odd or even. Consider the following definition.

An **even** number is one which can be written in the form $2 \times n$ where n is any integer.

For example, consider the following statement.

B 6 is even.

Most people on the street would agree that this is statement is true. Although it is trivial, the statement actually requires proof – there is no definition (at least in this document) which specifically says that "6 is even". But from the definitions above, we can construct the following proof.

Proof of B

Since we can write 6 as 2×3 and 3 is an integer, from the definition of "even", we can conclude that 6 is an even number. The proof is complete.

We can now be certain that statement B is true.

This proof is an example of **deductive reasoning**. We began at some starting point and arrived at a conclusion. The starting point may be clearly identified by saying "assume that…" or "suppose that…", but assumptions may not always be clearly indicated (as in the proof of B above).

The assumptions take the form of statements or definitions, and conclusions take the form of statements.

In the proof of B, our assumption was the definition of "even" and our conclusion was the statement "6 is even". We arrived at the conclusion based on our assumption.

This was a simple example to demonstrate how we use a definition to prove a statement. In practice, statements are much more complicated – they may rely on several definitions and the deductions may require much more than two lines of working.

EXAMPLES

1. Give a definition of what it means for a number to be "odd".

Notice that odd numbers are of the form $(\text{even number} + 1)$, e.g. $1 = 0 + 1$, $3 = 2 + 1$, and so on.

We can therefore define an odd number to be one which can be written in the form $2n + 1$ where n is any integer.

Alternatively, we could say that any integer which is not even is odd. In fact these definitions are equivalent.

2. Prove the following statements.

   (a) 9 is odd;

   (b) –30 is even.

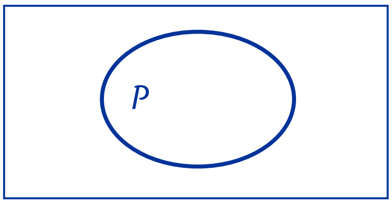
(a) Since $9 = 2 \times 4 + 1$, it is an odd number.
(b) Since $-30 = 2 \times -16 + 1$, it is an even number.

Euler diagrams

Before we begin making more complicated deductions, we will consider Euler (pronounced "oiler") diagrams. These will help us later.

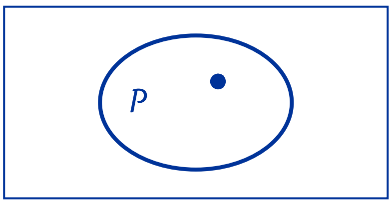
In an Euler diagram, we represent a statement with a closed shape, usually an ellipse.

For example, we might represent a statement P as follows.



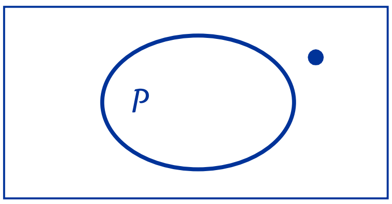
Representation of a statement P using an Euler diagram.

When statement P is true, we show this by placing a dot within the shape which represents statement P.



P is true.

When P is false, we put a dot outside the shape.



P is false.

The rectangle enclosing the shape represents the "universe" in which the statement exists. When placing a dot, it must be inside the rectangle.

Any dot is either inside the shape or outside it – a statement must either be true or false (but not both at once).

The value of this representation is probably not apparent at this stage, but as we look at more complicated statements, Euler diagrams will help us a great deal.

3 Logical Connectives

In the previous section, we defined that a statement is a sentence (or clause) which must be either true or false (but not both at once).

In English, given two sentences, we can construct other ones by "joining" them together. For example, consider the statements:

I will go to school today.	I will go to school tomorrow.

We can combine these into one sentence in several ways. We could say

I will go to school today **and** I will go to school tomorrow.

I will go to school today **or** I will go to school tomorrow.

If I will go to school today **then** I will go to school tomorrow.

These all mean different things, as we will discuss in this section.

We can also create a new sentence from a single sentence by adding a word which alters its meaning. Usually we use the word "not".

I will go to school today.	I will *not* go to school tomorrow.

From this, we can again construct a number of larger sentences.

I will go to school today **but** I will *not* go to school tomorrow.

I will go to school today **or** I will *not* go to school tomorrow.

If I will go to school today **then** I will *not* go to school tomorrow.

Note that in English the word "but" serves the same purpose as the word "and" – it merely serves to add emphasis. The meaning of a sentence does not change if we replace one with the other.

In mathematics we can also construct larger statements from smaller ones, as we will see in this section.

Negation

Consider the statements:

A I will go to school today.

B I will not go to school today.

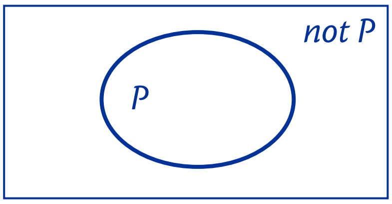
When we have two statements related like this, we say that B is the **negation** of A. Also, A is the negation of B.

In general, the negation of a statement P is written as

not P.

So in the above example B, is the same as *not* A.

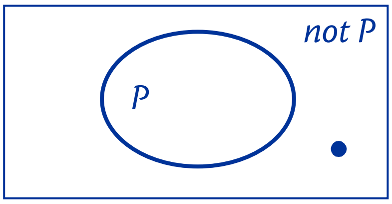
The Euler diagram for two statements P and *not* P is shown below.



Representation of a statement P and its negation *not* P.

Here, the area outside the shape labelled P represents the statement *not* P.

Since we put a dot inside P to mean that P is true, placing a dot anywhere outside P means that *not* P is true. Hence when P is false, *not* P is true and when P is true, *not* P is false.



P is false so *not* P is true.

In the examples above, if A is true then we can see that B must be false.

A statement P and its negation *not* P always have different truth values – when one is true, the other is false.

Note

The statement

I will go to school tomorrow.

is not the negation of the statement

I will go to school today.

In fact (other than being about going to school) these statements are completely unrelated. They could both be true, both be false, or both have different truth values at a particular time.

Conjunction

Larger statements can be constructed from smaller ones using a **conjunction**. In English, this corresponds to the use of the words "and" or "but" to join sentences together.

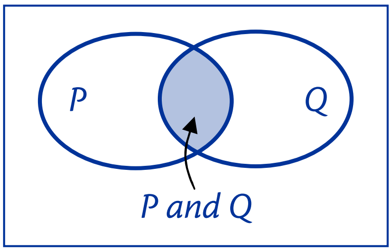
In general, given statements P and Q, their conjunction is written as

$$P \text{ and } Q.$$

For example, the following statement is a conjunction.

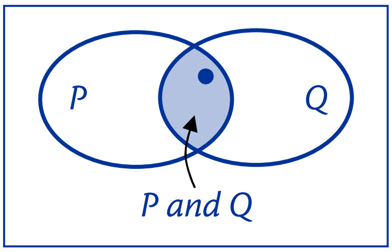
A I will go to school today and I will go to school tomorrow.

The Euler diagram for the statement P *and* Q is shown below.



Representation of the statement
P *and* Q.

Hence if both P and Q are true (we place a single dot inside both P and Q) then P *and* Q is also true (the dot is also inside the shaded area representing P *and* Q).



P is true and Q is true, so
P *and* Q is true.

Note that this is the only way in which P *and* Q can be true. Note also, that if P *and* Q is true, then P is true and Q is true.

Hence for statement A above to be true, I have to go to school today **and** go to school tomorrow. If I do not do either one of these things, then A is false.

EXAMPLES

Determine and explain the truth value of each of the following statements.

(a) 5 is even and –3 is even;

Since "5 is even" is false and "–3 is even" is false, the statement

"5 is even *and* –3 is even"

is false.

(b) 16 is even and 13 is odd;

Since "16 is even" is true and "13 is odd" is true, the statement

"16 is even *and* 13 is odd"

is true.

(c) –8 is odd and 9 is odd.

Since "–8 is odd" is false and "9 is odd" is true, the statement

"–8 is odd *and* 9 is odd"

is false.

Disjunction

Another way to construct larger statements from smaller ones is using a **disjunction**. In English, this corresponds to the use of the word "or" to join sentences together.

In general, given statements P and Q, their disjunction is written as

$$P \ \textit{or} \ Q.$$

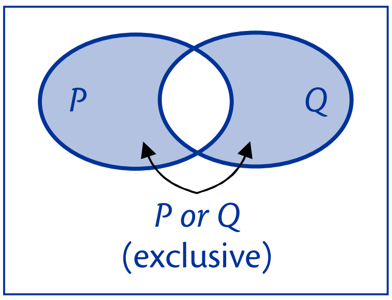
For example, the following statement is a disjunction.

A I will go to school today or I will go to school tomorrow.

Reading statement A, you might think that I will be in school on only one day: either today or tomorrow. This is known as an **exclusive** "or" and is common in English language.

This is unfortunate, because in mathematics we take A to mean: either I will go to school today, or I will go to school tomorrow, or both. This is known as an **inclusive** "or".

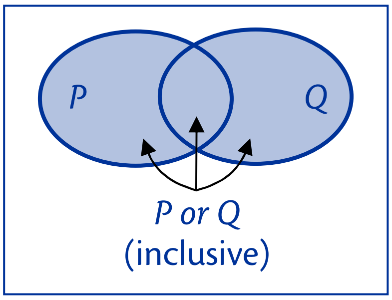
Looking at the Euler diagrams for the exclusive and inclusive "or" allows us to see the difference clearly.



Representation of the statement *P or Q* using exclusive "or".

Hence if both P and Q are true, the exclusive "or" states that P *or* Q is false. In other words, for P *or* Q (exclusive) to be true, P and Q must have different truth values (one must be true and the other false).

In contrast, the inclusive "or" has the following Euler diagram.



Representation of the statement P *or* Q using inclusive "or".

If both P and Q are true, we consider P *or* Q (inclusive) to be true also.

In mathematics (unless otherwise stated) it is assumed that a disjunction is **inclusive**.

EXAMPLES

Determine and explain the truth value of each of the following statements.

(a) 5 is even or –3 is even;

Since "5 is even" is false and "–3 is even" is false, the statement

"5 is even *or* –3 is even"

is false.

(b) 16 is even or 13 is odd;

Since "16 is even" is true and "13 is odd" is true, the statement

"16 is even *or* 13 is odd"

is true.

Note that the statement is false if we are using the exclusive "or".

(c) –8 is odd or 9 is odd.

Since "–8 is odd" is false and "9 is odd" is true, the statement

"–8 is odd *or* 9 is odd"

is true.

Note that this statement is true when using either the inclusive or exclusive "or".

Conditional

Consider the following statements:

If today is Saturday, then I will not go to school today.

and

For numbers a and b, if $a = b$ then $a^2 = b^2$.

These are called **conditional** statements.

In general, a conditional statement has the form

$$\textit{if } P \textit{ then } Q$$

where P and Q are statements.

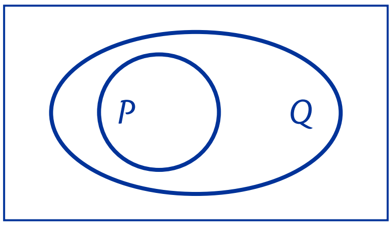
Note that in English, there are several ways of expressing a conditional statement. The following statements mean the same thing:

If today is Saturday, then I will not go to school today.

I will not go to school today if today is Saturday.

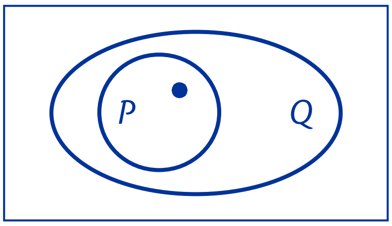
In order to simplify how we express conditional statements, in mathematics we often write $P \Rightarrow Q$ to mean *if P then Q*. This is read "*P* implies *Q*".

The conditional statement $P \Rightarrow Q$ tells us that if P is true then Q is also true. The Euler diagram representing a conditional statement looks as follows.



Representation of the statement *if P then Q*.

So if P is true (we put a dot inside P), then Q will also be true (the dot will automatically be inside Q).



P being true forces *Q* to be true.

It is impossible for Q to be false if P is true – we cannot place a dot inside P but outside Q.

For example, consider the statement

> For numbers a and b, if $a = b$ then $a^2 = b^2$.

Proof of statement

We start with "$a = b$" and we want to show that this leads to "$a^2 = b^2$".

First we take $a = b$ and multiply both sides of the equation by a:

$$a = b$$
$$a \times a = b \times a$$
$$a^2 = ab$$

This tells us that $a = b \Rightarrow a^2 = ab$ (i.e. if $a = b$ then $a^2 = ab$).

Then we do a similar thing, this time multiplying by b:

$$a = b$$
$$a \times b = b \times b$$
$$ab = b^2$$

This tells us that $a = b \Rightarrow ab = b^2$.

Now we know that $ab = a^2$ and $ab = b^2$, hence $a^2 = b^2$.

Therefore assuming that $a = b$ allows us to deduce that $a^2 = b^2$. The proof is complete.

We will discuss this statement further once we have met the "converse" of a conditional statement.

Note

The proof of the statement may seem too detailed or formal, but this is just to show that we should always be careful when constructing a proof.

Converse

With conjunctions and disjunctions the order of statements does not matter, e.g. P *and* Q and Q *and* P always have the same truth value. This makes sense since, for example the statements

> I will go to school today and I will go to school tomorrow.

and

> I will go to school tomorrow and I will go to school today.

mean exactly the same thing – they are equivalent statements.

However, the order of statements in a conditional statement *does* matter.

In general, the **converse** of the statement *if* P *then* Q is

$$\textit{if } Q \textit{ then } P$$

where P and Q are statements.

Given that a conditional statement $P \Rightarrow Q$ is true, we cannot assume that the converse $Q \Rightarrow P$ is true.

To illustrate this, consider the following statement.

A If today is Saturday, then I will not go to school today.

The converse of A is:

B If I will not go to school today, then today is Saturday.

Statement B is equivalent to saying: the only day I do not go to school is Saturday. This is false.

On the other hand, the converse may indeed be true. The statement

C If today is the 25th of December, then today is Christmas Day.

has converse

D If today is Christmas Day, then today is the 25th of December.

Suppose C is true. Then the converse D is also true.

Hence the truth of a conditional statement tells us nothing about the truth of its converse.

Equivalence

Consider the statement

If today is the 15th of April then today is Euler's birthday and if today is Euler's birthday then today is the 15th of April.

If this statement is true then it means that

A If today is the 15th of April then today is Euler's birthday.

and its converse

B If today is Euler's birthday then today is the 15th of April.

are true. Remember that this means more than just saying A is true on its own as the converse is not then automatically true.

In general, P and Q are said to be **equivalent** when

$$P \Rightarrow Q \ \ and\ Q \Rightarrow P$$

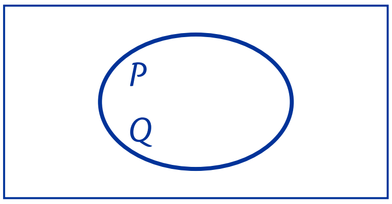
where P and Q are statements.

Symbolically, we write $P \Leftrightarrow Q$ to mean $P \Rightarrow Q \ \ and\ Q \Rightarrow P$. This is usually read as "P if, and only if Q". Furthermore we often shorten "if, and only if" to "iff" and so the above statement would become:

Today is the 15th of April iff today is Euler's birthday.

The statement $P \Leftrightarrow Q$ means that P and Q are equivalent. This means that they always have the same truth value – if one is true then the other is true, and if one is false then the other is false.

In terms of Euler diagrams, this means that P and Q share the same shape in the diagram.



Representation of the statement *P iff Q*.

A dot placed anywhere will either be inside both P and Q or outside both P and Q.

For example, consider the statement

For *positive* numbers a and b, $a = b$ iff $a^2 = b^2$.

Let's introduce the following labels:

C $a = b$.

D $a^2 = b^2$.

So the statement is saying that, for positive numbers $C \Leftrightarrow D$, i.e. $C \Rightarrow D$ and $D \Rightarrow C$.

Proving that $C \Leftrightarrow D$ means proving that $C \Rightarrow D$ and $D \Rightarrow C$.

We have already proven that $C \Rightarrow D$ for all numbers a and b and so if we assume that a and b are positive, the same proof is valid.

To prove that $D \Rightarrow C$, we can proceed as follows.

$$a^2 = b^2$$
$$\sqrt{a^2} = \sqrt{b^2}$$
$$a = b \qquad \text{since } a \text{ and } b \text{ are positive}$$

Since we have proven $C \Rightarrow D$ and $D \Rightarrow C$, we know that $C \Leftrightarrow D$.

Note that if a and b were allowed to be negative numbers then the proof of $D \Rightarrow C$ does not hold since, for example, if $a = -3$:

$$\sqrt{a^2} = \sqrt{(-3)^2} = \sqrt{9} = 3 \neq a.$$

As was said in the introduction, a mathematical proof "must be a correct logical argument". But how do we decide what is "correct"?

The deductions we have discussed in this section are considered "correct" by mathematicians. Based on the assumption that these deductions are valid, we can construct proofs of true mathematical statements.

4 Quantifiers – the need for more descriptive power

Consider the following statement:

A x is an even number.

The truth value of A depends on the value of x. For example,

if $x = 32$ then A is true;
if $x = 17$ then A is false.

We cannot determine the truth value of A without knowing something about the value of x.

In statement A, the letter "x" is called a **variable** because its value can change. Letters can also be used to stand for **constants**, e.g. π. The value of a constant is fixed.

In mathematics, we usually want to make true statements so that we can use these as rules or theorems. Variables therefore cause a problem since a statement can be true in one instance and false in another.

Therefore we need additional tools to make sure our statements are clear – these are called **quantifiers**.

Note

Variables need not take the value of a number. For example, in the statement

y can play the piano.

the variable y might take the value of a person's name.

Usually the type of a variable is obvious from its context. Unless otherwise stated, we will assume that variables can take any *real* value. By this we mean any point on the number line.

To ensure that statements say exactly what we intend and that others can understand this, it does no harm to always explicitly state the type of a variable and any restrictions on its value.

In more advanced mathematics (when we may talk about things other than numbers) it is important to be clear about the type of a variable.

Existential quantifier

Consider, again, statement A from above.

A x is an even number.

Suppose that, in order to make sure the statement is true, we modify A to be very vague:

B There is some value of x for which x is an even number.

Don't be confused by this; all B is saying is that there is at least one even number.

The phrase "there is some" is called the **existential quantifier** – it is used to state that something "exists". (The symbol "$\exists$" is sometimes used in place of the phrase "there is some".)

We can easily prove B as follows. Let $x = 2$. Then x is an even number. Since we have found a value for x which is even, we have proven B.

Note that B makes no claim about the exact number of cases in which x is even; it just says that there is at least one value of x which is even.

To prove a statement with only an existential quantifier, it is sufficient to show that the statement is true in one relevant case.

Universal quantifier

Sometimes we might want to make a statement which is true in all cases. For example, we have already proven the statement

For numbers a and b, if $a = b$ then $a^2 = b^2$.

Another way of stating this is to say that

C For *all* numbers a and b, if $a = b$ then $a^2 = b^2$.

The phrase "for all" is called the universal quantifier – it is used to state that something is always true. (The symbol "$\forall$" is sometimes used in place of the phrase "for all".)

Note

There is an explicit restriction on what a and b can be in this statement; it says "for any *numbers*". Hence the universal quantifier in this statement only applies to numbers – it doesn't say anything about a^2 being equal to b^2 if a and b are not numbers.

Statements with a universal quantifier are (in general) more complicated to prove than those with only an existential quantifier.

For example, we can verify that C is true in any number of cases by choosing specific values for a and b:

- Let $a = b = 1$ then $a^2 = 1$ and $b^2 = 1$ so $a^2 = b^2$.
- Let $a = b = \frac{1}{2}$ then $a^2 = \frac{1}{4}$ and $b^2 = \frac{1}{4}$ so $a^2 = b^2$.
- Let $a = b = -13$ then $a^2 = 169$ and $b^2 = 169$ so $a^2 = b^2$.

We could go on doing this forever and there would still be some numbers we wouldn't have considered.

Given statements that apply to an infinite number of cases, we <u>cannot</u> prove the statement just by considering examples – there could always be one case we have not considered where the statement is false.

We must therefore prove that C is true without assuming anything more than that a and b are numbers and that $a = b$; we did this on Page 11.

5 Proof Techniques

Building on the deductions we discussed in Section 3, we will now look at a number of techniques which are used to prove mathematical statements.

Note that there may be several different ways of proving a statement. Each proof is just as valid as any other, as long as it is correct.

In this section, we will use the following definition.

A positive integer n is said to be **prime** if n has exactly two distinct positive integer factors: n and 1.

Disproving statements

Before we look at proving statements, we will first consider how we can disprove a statement.

Consider the following statement.

For any numbers a and b, if $a^2 = b^2$ then $a = b$.

Statements like this, with a universal quantifier, can be disproved if we find a single example in which the statement is false.

Since the statement says that a and b can be any numbers, if we find a values for a and b for which the statement is false, then it is not true for any numbers a and b.

To disprove the above statement, we can take $a = -3$ and $b = 3$. Then $a^2 = 9$ and $b^2 = 9$, therefore $a^2 = b^2$. But this does not mean $a = b$ because we know $a \neq b$.

This technique is called providing a "counterexample" to the statement.

EXAMPLES

1. Disprove the statement "all odd numbers between 2 and 14 are prime".

   We can provide the following counterexample.

   Consider the number 9 which is between 2 and 14 but is not prime since it can be written as 3×3.

   Hence not all numbers between 2 and 14 are prime.

2. Disprove the statement "for all real x, $x^2 + x > 0$".

   Let $x = -1$ then $x^2 + x = 1 - 1 = 0$.

   Hence $x^2 + x$ is not greater than 0 for all real x.

Proof by exhaustion

Sometimes a statement will refer only to a closed "universe". For example the statement

> The numbers 2, 3, 5 and 7 are prime.

refers only to the numbers 2, 3, 5 and 7.

We can prove this statement by verifying that each of the numbers 2, 3, 5 and 7 is prime.

In reality, statements of this kind are often proven by a computer, since a machine can carry out computations much more quickly than a person.

EXAMPLES

1. Prove that there are two prime numbers between 20 and 30.

   23 and 29 are two prime numbers between 20 and 30. The proof is complete.

   Is the statement "there are two prime numbers between 20 and 40" true?

2. Prove that for $x = 2, 3, 4$, $x^2 - 3$ is between 0 and 14.

   Consider each case separately:

   if $x = 2$ then $x^2 - 3 = 4 - 3 = 1$;
   if $x = 3$ then $x^2 - 3 = 9 - 3 = 6$ and
   if $x = 4$ then $x^2 - 3 = 16 - 3 = 13$.

   Hence $x^2 - 3$ is between 0 and 14 for $x = 2,3,4$. The proof is complete.

Direct proof

It may sometimes be reasonably simple to prove a statement directly using deductive reasoning.

The most useful statements in mathematics are usually conditional (or equivalence) statements. Proving a conditional statement $P \Rightarrow Q$ directly means assuming that P is true, and showing that this leads to us saying that Q must also be true.

We have already seen an example of this in the proof that $a = b \;\Rightarrow\; a^2 = b^2$.

EXAMPLES

1. Prove that if a and b are even numbers then $a \times b$ is also even.

Let the two even numbers be a and b, so that we have

$$a = 2k \text{ and}$$
$$b = 2l$$

where k and l are integers. Then their product $a \times b$ can be expressed as

$$\begin{aligned} a \times b &= 2k \times 2l \\ &= 4kl \\ &= 2 \times (2kl). \end{aligned}$$

Since $2kl$ is an integer, $a \times b$ has the form of an even number. Hence the product of two even numbers is even.

2. Prove that if a, b, c and d are numbers where $b,\ c,\ d \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

Suppose a, b, c and d are numbers and $b,\ c,\ d \neq 0$. Then

$$\begin{aligned} \frac{a}{b} \div \frac{c}{d} &= \frac{\frac{a}{b}}{\frac{c}{d}} \\ &= \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}} \qquad \text{multiplying top and bottom by } \frac{d}{c} \\ &= \frac{a}{b} \times \frac{d}{c}. \end{aligned}$$

The proof is complete.

3. Prove that if n is any positive integer then the number n^2+n is even.

Suppose n is a positive integer. We know that $n^2+n=n(n+1)$ by factorising.

Now n is either even or odd. We can consider these two cases separately.

Suppose n is even, so we know that $n=2k$ for some positive integer k.

Then $n+1=2k+1$. Hence $n+1$ is odd. Then we see that

$$\begin{aligned} n^2+n &= n(n+1) \\ &= (2k)(2k+1) \\ &= 2\times k(2k+1) \end{aligned}$$

and so n^2+n is even since it is of the form $(2\times \text{an integer})$.

Now suppose n is odd, so we know that $n=2k+1$ for some positive integer k.

Then
$$\begin{aligned} n+1 &= 2k+1+1 \\ &= 2k+2 \\ &= 2\times(k+1). \end{aligned}$$

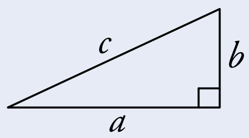
Hence $n+1$ is even. Then we see that

$$\begin{aligned} n^2+n &= n(n+1) \\ &= (2k+1)\times 2(k+1) \\ &= 2\times(2k+1)(k+1) \end{aligned}$$

and so n^2+n is even since it is of the form $(2\times \text{an integer})$.

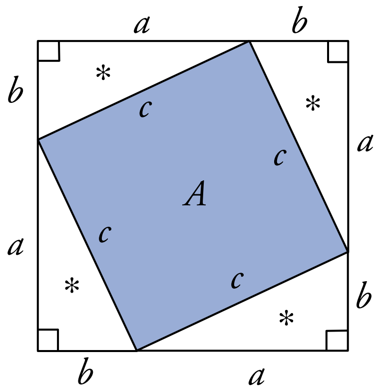
The proof is complete.

4. Prove that, for the triangle shown below,



the square of the hypotenuse is equal to the sum of the squares of the other two sides, i.e. that $c^2 = a^2 + b^2$.

Consider the following arrangement of four copies of the above triangle.



Now, the shaded shape is a square (you can prove this yourself by thinking about the angles around its vertices).

The area, A square units, of the shaded square is c^2, i.e. $A = c^2$.

But we can also find the area of the shaded square in a different way.

If we work out the area of the surrounding square (with side $a+b$), then take off the area of the four triangles marked $*$, this will give us another expression for the area of the shaded square.

The area of the large surrounding square, in square units, is

$$(a+b)^2 = a^2 + 2ab + b^2 .$$

Each triangle marked $*$ has area $\frac{1}{2}ab$ square units, and so the four triangles have a combined area of

$$4 \times \tfrac{1}{2}ab = 2ab .$$

To get A, the area of the shaded square, we can subtract the area of the four triangles marked $*$ from the area of the surrounding square. Hence

$$\begin{aligned} A &= a^2 + 2ab + b^2 - 2ab \\ &= a^2 + b^2 . \end{aligned}$$

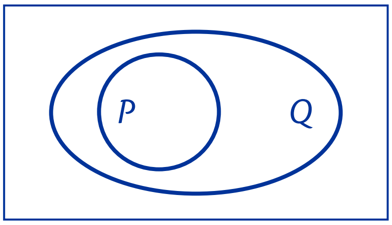
Therefore $A = c^2 = a^2 + b^2$. The proof is complete.

Proof by contradiction

Suppose we have the conditional statement $P \Rightarrow Q$. Remember, this means:

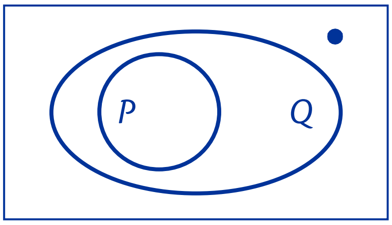
"If P then Q".

Recall that the Euler diagram for $P \Rightarrow Q$ looks as follows.



Representation of the statement $P \Rightarrow Q$.

For Q to be false (i.e. to place a dot outside Q), P must also be false (i.e. the dot must also be outside P).



If Q is false, then P is false.

Now suppose that we have a statement Q which is always false, this is called an **absurdity** (i.e. wherever we place a dot in the Euler diagram, it must be outside Q). It follows that P must always be false also – we can never place a dot inside P if we are not allowed inside Q.

To prove that a statement P is false, we can assume P is true and show that

$$P \Rightarrow F$$

where F is any statement which is false (i.e. an absurdity).

EXAMPLES

1. Disprove the statement "there is a number which is both odd and even".

Suppose that such a number does exist, call it a. Then

$a = 2k$ (i.e. a is even) and
$a = 2l + 1$ (i.e. a is odd)

where k and l are integers. This means that

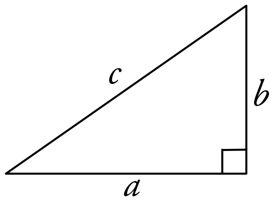
$$\begin{aligned} 2k &= 2l + 1 \\ 2k - 2l &= 1 \\ 2(k - l) &= 1 \\ k - l &= \tfrac{1}{2}. \end{aligned}$$

This can never be true because if k and l are integers then $k-l$ must also be an integer. But $\frac{1}{2}$ is not an integer, therefore the statement $k-l=\frac{1}{2}$ is absurd – it is always false.

Therefore we have shown that assuming the statement leads to an absurdity and hence the statement itself must be false.

2. Disprove the statement "there is a right-angled triangle whose hypotenuse is longer than the sum of the other two sides".

Suppose that such a triangle does exist and label the sides as follows.



So the statement says that $c > a+b$. Squaring both sides gives

$$c^2 > (a+b)^2$$
$$c^2 > a^2 + 2ab + b^2.$$

Using Pythagoras' Theorem, we know that $c^2 = a^2 + b^2$ and so we have

$$a^2 + b^2 > a^2 + 2ab + b^2$$
$$2ab < 0.$$

This is saying that $2ab$ is negative. This is absurd because a and b are positive (since they are lengths) which means $2ab$ cannot be negative.

Therefore assuming that the statement is true leads to an absurdity and so the statement itself must be false. The proof is complete.

We will now discuss a more powerful use of the above technique. Suppose that we want to prove that a statement P is true.

If we can prove that *not* P is false, then we can deduce that P is true (since the negation of a statement has the opposite truth value).

To prove that a statement P is true, we can assume that *not* P is true and show that

$$\textit{not } P \Rightarrow F$$

where F is any statement which we know is false.

That is, assuming P is false leads to an absurdity. So the assumption that "P is false" is false, hence P must be true.

Note

We have used the word "absurdity" to describe a statement which is always false. These absurdities really fall into two separate categories.

A statement can be absurd because it says something that we know to be false based on our knowledge of mathematics. For example, $1=0$ is absurd.

On the other hand, a statement that goes against any of the assumptions we make is also false. For instance, in Example 1 above, the statement $k-l=\frac{1}{2}$ is absurd because it *contradicted* our assumption that k and l were integers.

Proving a statement by assuming its negation and arriving at either of these types of absurdity is called a "proof by contradiction".

EXAMPLES

1. Prove that there is no largest integer.

First we can label the statement, call it L.

Based on the above argument we want to show that $not\ L \Rightarrow F$.

The negation of L is *not* L: "there is a largest integer." We can assume this and show that this leads to a false statement.

Let k be this largest integer and so *not* L is equivalent to saying that all integers are less than, or equal to, k.

But $k+1$ is also an integer and it is greater than k. So based on our assumption, k is not the largest integer. This is false because we assumed that k was the largest integer!

Hence *not* L leads to a false statement (given our assumptions) and so *not* L must be false. Therefore L must be true. The proof is complete.

2. Prove that there are no integer solutions to the equation $x^2-4y=3$.

We will assume that the statement is false and show that this leads to a contradiction.

Assume that there *are* two integers x and y such that $x^2-4y=3$. Now if x is an integer, then it is either even or odd. We will consider these two cases separately and show that we reach a contradiction in both cases.

Suppose x is even and so let $x = 2k$ where k is an integer. We have

$$\begin{aligned} x^2 - 4y &= 3 \\ (2k)^2 - 4y &= 3 \\ 4k^2 - 4y &= 3 \\ 4y &= 4k^2 - 3 \\ y &= k^2 - \tfrac{3}{4}. \end{aligned}$$

But if k is an integer, then so is k^2 and so $y = k^2 - \frac{3}{4}$ is not an integer.

Hence assuming that x is an even integer and that y is an integer leads us to say that y is not an integer. This is a contradiction.

Now consider the case where x is odd and so let $x = 2l + 1$ where l is an integer. We have

$$\begin{aligned} x^2 - 4y &= 3 \\ (2l+1)^2 - 4y &= 3 \\ 4l^2 + 4l + 1 - 4y &= 3 \\ 4y &= 4l^2 + 4l - 2 \\ y &= l^2 + l - \tfrac{1}{2}. \end{aligned}$$

But if l is an integer, then so is l^2 and so is $l^2 + l$. This means $y = l^2 + l - \frac{1}{2}$ is not an integer – because $(\text{an integer}) - \frac{1}{2}$ cannot be an integer.

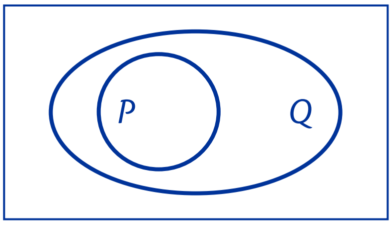
Hence assuming that x is an odd integer and that y is an integer leads us to say that y is not an integer. This is a contradiction.

Because x must be either even or odd, and we have shown that in both cases there is no integer solution for y, then we can conclude that no integer solutions exist.

Proving the contrapositive

The statements $P \Rightarrow Q$ and *not* $Q \Rightarrow$ *not* P are equivalent. To see this, consider the Euler diagrams representing these two statements.

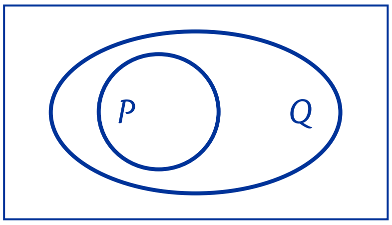
Recall the following Euler diagram.



Euler diagram for $P \Rightarrow Q$.

The statement $P \Rightarrow Q$ means that if we place a dot inside P then it is also inside Q.

Surely this means the same thing as saying: if a dot is outside Q then it must be outside P. If we draw the Euler diagram for this, then the shape for P must be inside the shape for Q, giving the same Euler diagram as above.



Euler diagram for *not* $Q \Rightarrow$ *not* P.

So suppose we want to prove a conditional statement $P \Rightarrow Q$. To do this, we can prove the statement *not* $Q \Rightarrow$ *not* P. Since these two statements are equivalent, they have the same truth value – if one is true then the other is true.

EXAMPLES

1. Prove that if a^2 is an even number then a is even.

Let A be the statement "a^2 is even" and B be the statement "a is even".

We will prove that $A \Rightarrow B$ by proving its contrapositive *not* $B \Rightarrow$ *not* A. The contrapositive is the statement "if a is not even then a^2 is not even". This means "if a is odd then a^2 is odd". We can prove this directly.

Suppose a is odd, then we can write $a = 2k+1$ where k is some integer. It follows that

$$\begin{aligned} a^2 &= (2k+1)(2k+1) \\ &= (4k^2+2k+2k)+1 \\ &= 2(2k^2+2k)+1. \end{aligned}$$

Now since k is an integer, it follows that $(2k^2 + 2k)$ is an integer. So a^2 can be written as $(2\times\text{an integer}+1)$ and hence a^2 is odd.

Since we have proven the contrapositive of the statement, and this is equivalent to the statement itself, the proof is complete.

2. Suppose that a, b and c are numbers and that $a > b$.
   Prove that if $ac \leq bc$ then $c \leq 0$.

Let A be the statement "$ac \leq bc$" and B be the statement "$c \leq 0$".

To prove $A \Rightarrow B$, we will prove the contrapositive *not* $B \Rightarrow$ *not* A. The contrapositive is "if $c > 0$ then $ac > bc$". We can prove this directly. Suppose $a > b$ and $c > 0$:

$$a > b$$
$$ac > bc \qquad \text{since } c > 0.$$

Hence we have proven the contrapositive *not* $B \Rightarrow$ *not* A. Since this is equivalent to the statement $A \Rightarrow B$, the proof is complete.

Proving equivalences

We have already met and proven some conditional statements, i.e. ones of the form $P \Rightarrow Q$.

Remember, saying that two statements P and Q are equivalent (written $P \Leftrightarrow Q$) means that $P \Rightarrow Q$ and $Q \Rightarrow P$. Hence proving $P \Leftrightarrow Q$ means proving $P \Rightarrow Q$ and $Q \Rightarrow P$.

In some cases, it is simplest to prove $P \Rightarrow Q$ and $Q \Rightarrow P$ separately then conclude that $P \Leftrightarrow Q$.

EXAMPLES

1. Suppose $n = 10a + b$ where a and b are positive integers. Prove that n is a multiple of 9 iff $(a+b)$ is a multiple of 9.

Notice that a positive integer being multiple of 9 means that it can be written in the form $9\times(\text{an integer})$. So the above statement is saying:

$$n = 9k \quad \Leftrightarrow \quad (a+b) = 9l.$$

First we will prove that $n = 9k \;\Rightarrow\; (a+b) = 9l$.

Suppose $n = 9k$. Then

$$\begin{aligned} 9k &= 10a + b \\ &= 9a + (a + b) \\ a + b &= 9k - 9a \\ &= 9(k - a). \end{aligned}$$

Hence $a + b$ is a multiple of 9.

Now we prove that $(a+b) = 9l \Rightarrow n = 9k$. Suppose $(a+b) = 9l$. Then

$$\begin{aligned} n &= 10a + b \\ &= 9a + (a + b) \\ &= 9a + 9l \\ &= 9(a + l). \end{aligned}$$

Hence n is a multiple of 9. The proof is complete.

2. Prove that a positive integer n is divisible by 6 iff n is divisible by 2 and 3.

First we will prove that if n is a multiple of 6 then n is a multiple of 2 and 3.

Suppose n is a multiple of 6, i.e. $n = 6k$ when k is an integer. Then

$$\begin{aligned} n &= 2 \times (3k) \quad && \text{hence a multiple of 2} \\ &= 3 \times (2k) && \text{hence a multiple of 3.} \end{aligned}$$

Now we prove that if n is a multiple of 2 and 3 then n is a multiple of 6.

Suppose:

- n is a multiple of 2, i.e. $n = 2l$
- n is a multiple of 3, i.e. $n = 2m$

for integers l and m.

$$\begin{aligned} \text{Now } n &= 3n - 2n \\ &= 3(2l) - 2(3m) \\ &= 6l - 6m \\ &= 6(l - m). \end{aligned}$$

Hence n is a multiple of 6. The proof is complete.

Sometimes we can prove the equivalence directly.

EXAMPLES

3. Prove that $a < b \iff \dfrac{a+b}{2} < b$.

Suppose $a < b$. Then

$$\begin{aligned} a < b &\iff a+b < b+b && \text{adding } b \text{ to both sides} \\ &\iff a+b < 2b \\ &\iff \frac{a+b}{2} < b && \text{dividing through by 2.} \end{aligned}$$

Each line in this proof is equivalent, i.e. from any line we can get to any other using correct deductions. Therefore the proof is complete.

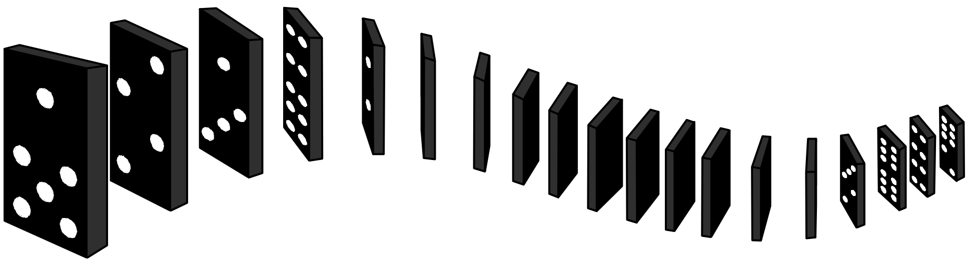
When proving equivalences in this way we must be careful that each line in the proof is equivalent, i.e. we can logically go from the first line to the last and from the last line to the first.

Proof by induction

This technique is common when proving statements about the natural numbers, i.e. the numbers 1, 2, 3, ….

To help us understand the technique, consider the following analogy.

Imagine an infinitely long line of dominoes.



Suppose we concentrate only on the *first* domino in the line. Then we topple it without thinking about what happens to any of the other dominoes.

Assuming that the dominoes are close enough together, we know that if we topple any domino in the line then the next one will topple.

Since we have toppled the first domino and we assumed that *any* domino being toppled causes the next to topple, we can conclude that *all* dominoes will topple.

In practice, this is true since we topple the first domino and this will topple the second, which will topple the third, which will topple the fourth, and so on. Hence all the dominoes will fall.

Now suppose that instead of a line of dominoes we have a mathematical statement which we wish to prove for all natural numbers. Each domino represents the statement for a single natural number.

For example, consider the statement:

If n is a natural number, then n^2+n is even.

Think of this as the line of dominoes. The kth domino in the line represents the statement "k^2+k is even".

Remember that we say that a number is "even" if, and only if it can be written in the form $2m$ where m is any integer.

Knocking over the first domino represents proving the statement for $n=1$.

For $n=1$:

$$n^2+n=1^2+1=2.$$

Since $2=2\times1$ (i.e. 2 is even), the statement has been proven for $n=1$.

Making sure that the dominoes are close enough together means the following here: if the statement is true for $n=k$ we must prove that it follows that it is true for $n=k+1$.

Assume the statement is true for $n=k$, i.e. $k^2+k=2m$ where m is an integer. Based on this assumption, we prove that the statement is true for the next integer, $k+1$.

For $n=k+1$:

$$\begin{aligned} n^2+n &= (k+1)^2+k+1 \\ &= k^2+2k+1+k+1 \\ &= (k^2+k)+2k+2 \\ &= 2m+2k+2 \qquad \text{based on the assumption that } k^2+k=2m \\ &= 2(m+k+1). \end{aligned}$$

Since m, k and 1 are integers, $(m+k+1)$ is an integer, hence $(k+1)^2+k+1$ can be written as $2\times\text{an integer}$. So assuming that the statement is true for $n=k$ means we can prove it for $n=k+1$.

Since we have shown that the statement is true for $n=1$, we now know that it will be true for $n=2$, and this will mean it is true for $n=3$, $n=4$, and so on forever.

In general, to prove a statement P for all positive integers n:

1. Show that P is true for $n=1$;
2. Assume that P is true for $n=k$ and use this assumption to prove that P is true for $n=k+1$.

The proof of the statement then follows by the "Principle of Mathematical Induction."

EXAMPLES

1. Prove that for any positive integer n, $7^n-1=6m$ where m is some positive integer. (This is saying that 7^n-1 is a multiple of 6 for all n.)

For $n=1$, $7^n-1=7^1-1=6$. Since $6=6\times 1$, the statement is true for $n=1$.

Assume that $7^k-1=6m$ for some positive integer m. Then $7^k=6m+1$.

For $n=k+1$:

$$\begin{aligned}
7^n-1&=7^{k+1}-1\\
&=7\times 7^k-1 && \text{since } 7\times 7^k=7^{k+1}\\
&=7\times(6m+1)-1 && \text{since } 7^k=6m+1 \text{ from our assumption}\\
&=42m+7-1\\
&=6(7m+1).
\end{aligned}$$

Hence assuming that the statement is true for $n=k$ means we can prove it for $n=k+1$. Since we proved the statement for $n=1$ the statement is true by the Principle of Mathematical Induction.

2. Prove that for all positive integers n, $1+2+3+\cdots+n=\frac{n(n+1)}{2}$.

For $n=1$:

$$\text{LHS}=1,$$

$$\text{RHS}=\frac{1(1+1)}{2}=1.$$

Since $\text{LHS}=\text{RHS}$, the statement is true for $n=1$.

Assume that the statement is true for $n=k$, i.e.

$$1+2+3+\cdots+k=\frac{k(k+1)}{2}.$$

For $n=k+1$:

$$\begin{aligned}\underbrace{(1+2+3+\cdots+k)}_{=\frac{k(k+1)}{2}\text{ from our assumption}}+(k+1)&=\frac{k(k+1)}{2}+(k+1)\\&=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}\\&=\frac{(k+1)(k+2)}{2}\\&=\frac{(k+1)((k+1)+1)}{2}.\end{aligned}$$

Hence assuming that the statement is true for $n=k$ means we can prove it for $n=k+1$. Since we proved the statement for $n=1$ the statement is true by the Principle of Mathematical Induction.

6 Keeping it Logical

Misuse of the "$\Rightarrow$" symbol

The conditional ($\Rightarrow$) symbol <u>must</u> only be used between two statements.

Remember that we consider a statement to be a sentence (or clause) which must be either true or false (but not both at once).

The conditional symbol does not mean the same thing as the equals symbol – they are certainly not interchangeable.

For example, "$4+5$" is not a statement – it cannot be assigned a truth value. As such, "$4+5 \Rightarrow 9$" demonstrates incorrect usage of the conditional symbol.

Instead, we should write "$4+5=9$." This *is* a statement – it reads as a sentence which can be assigned a truth value. This can then be used along with another statement to form a conditional, e.g. $4+5=9 \Rightarrow 4<9$.

A simple (but effective) check is that whenever you write the "$\Rightarrow$" symbol you should read it to yourself as "if… then…". For example, "if $4+5=9$ then $4<9$" reads as a sentence, whereas "if $4+5$ then 9" certainly does not!

This might seem obvious in this simple, isolated example but it is important to use the "$\Rightarrow$" symbol properly whenever you do mathematics.

When is a solution not a solution?

Consider the following solution to the equation $\sqrt{x}=2-x$ for $x>0$.

$$\begin{aligned}
\sqrt{x} &= 2-x \\
x &= (2-x)^2 \\
x &= 4-4x+x^2 \\
x^2-5x+4 &= 0 \\
(x-4)(x-1) &= 0
\end{aligned}$$

This suggests that $x=1$ and $x=4$ are solutions of the equation.

If we try $x=1$ in the equation, then all is well:

$$\begin{aligned}
\text{LHS} &= \sqrt{1} = 1 \\
\text{RHS} &= 2-1 = 1.
\end{aligned}$$

Since $\text{LHS}=\text{RHS}$, $x=1$ is a solution.

But trying $x=4$, we notice:

$$\begin{aligned}
\text{LHS} &= \sqrt{4} = 2 \\
\text{RHS} &= 2-4 = -2.
\end{aligned}$$

Now $\text{LHS} \neq \text{RHS}$, so $x=4$ is not a solution! What lead us to this "wrong" solution?

The problem becomes clear if we carefully keep track of the reasoning used in the solution.

Consider the same solution again, but this time notice the symbols used to relate each line to the one before.

$$\begin{array}{lrl} & \sqrt{x} = 2 - x & \quad (1) \\ \Rightarrow & x = (2-x)^2 & \quad (2) \\ \Leftrightarrow & x = 4 - 4x + x^2 & \\ \Leftrightarrow & x^2 - 5x + 4 = 0 & \\ \Leftrightarrow & (x-4)(x-1) = 0 & \end{array}$$

The source of the "error" is the statement from line (1) to (2):

$$\sqrt{x} = 2 - x \;\Rightarrow\; x = (2-x)^2.$$

This statement is true but the converse is not. While $x = (2-x)^2$ is true for $x = 4$, this does not mean that $\sqrt{x} = 2 - x$.

We have shown in Section 3 that $a = b \;\Rightarrow\; a^2 = b^2$ for any numbers *a* and *b*, but we showed that the converse is false.

Note that when solving an equation we are simply looking for values of the variable which mean that the left hand side is equal to the right hand side. So ending up with "wrong" solutions as we did above is certainly possible if there is an implication rather than an equivalence along the way.

When solving an equation, we should check that the "solutions" we derive are, in fact, solutions!