Advanced Higher

Mathematics

UNITS 1, 2 AND 3

Course Summary

HSN28000

Contents

Binomial Theorem and Partial Fractions

1 Binomial Theorem

$$^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

For $n \in \mathbb{N}$, the Binomial Theorem states:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n}y^n.$$

The rth term of $(x+y)^n$ is given by:

$$\binom{n}{r}x^{n-r}y^r.$$

2 Partial Fractions

$$\frac{f(x)}{(x-a)(x+b)} = \frac{A}{(x-a)} + \frac{B}{(x+b)}$$

$$\frac{f(x)}{(x-a)^2} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2}$$

$$\frac{f(x)}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$$

$$\frac{f(x)}{(x-a)(x-b)^2} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-b)^2}$$

$$\frac{f(x)}{(x-a)^3} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$$

$$\frac{f(x)}{(x-a)(x^2+bx+c)} = \frac{A}{(x-a)} + \frac{Bx+C}{(x^2+bx+c)}$$

If the numerator is of a higher (or equal) degree than the denominator, then algebraic long division should be used first to obtain a proper rational function.

Matrices

1 Gaussian Elimination

Elementary row operations are used to turn the matrix into upper-triangular form.

Geometric interpretation

- If one row's results are all zeros then the three planes do not intersect at a point but they intersect in a line.
- If one row has zeros in the x, y and z places but a non-zero number on the right of the line then there is no solution and the planes do not intersect.

Ill-conditioning

This occurs when a small difference in the coefficients of the equations causes a large change in the solution.

2 Matrix Algebra

Addition and Subtraction

Matrices can only be added or subtracted if they have the same order. Add or subtract entry-wise.

Multiplication by a Scalar

Multiply each element by the scalar.

Matrix Multiplication

Multiply row 1 in first matrix by column 1 in the second matrix, then multiply row 1 in first matrix by column 2 in second matrix etc.

EXAMPLE

$$\begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 0 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} (2\times 4)+(3\times 0)+(-1\times 1) & (2\times 1)+(3\times 2)+(-1\times 3) \\ (1\times 4)+(2\times 0)+(0\times 1) & (1\times 1)+(2\times 2)+(0\times 3) \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 5 \\ 4 & 5 \end{pmatrix}.$$

Determinants

The determinant of a 2×2 matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det A = ad-bc$.

The determinant of a 3×3 matrix $A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is

$$\begin{aligned}\det A &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix}+b\begin{vmatrix} d & f \\ g & i \end{vmatrix}+c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a\left(ei-fh\right)+b\left(di-fg\right)+c\left(dh-eg\right).\end{aligned}$$

Inverses

For an inverse of a matrix to exist the matrix must satisfy two conditions:

- the matrix must be square;
- the determinant of the matrix must be non-zero.

The inverse of a 2×2 matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1}=\dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

The inverse of a 3×3 matrix is found by starting with the augmented matrix $\left(A \mid I\right)$ and then using elementary row operations to turn it into $\left(I \mid A^{-1}\right)$.

Transpose of a Matrix

The transpose of a matrix is found by interchanging the rows and columns.

EXAMPLE

If $A=\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ then $A^{\mathrm{T}}=\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$.

3 Transformations of the Plane

Rotation

Rotation through angle θ in the anticlockwise direction about the origin is represented by $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

EXAMPLE

Rotation by $\frac{\pi}{2}$ is given by $\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Reflection

Reflection in the line which makes an angle of θ with the x-axis is given by $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$.

EXAMPLE

Reflection in the line $y = x$ is given by

$$\begin{pmatrix} \cos\left(2\times\frac{\pi}{4}\right) & \sin\left(2\times\frac{\pi}{4}\right) \\ \sin\left(2\times\frac{\pi}{4}\right) & -\cos\left(2\times\frac{\pi}{4}\right) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Scaling

Scaling is given by $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$.

This has the effect of multiplying the x-coordinate by λ and the y-coordinate by μ.

Sequences and Series

1 Arithmetic Sequences

An arithmetic sequence has the form

$$a,\ a+d,\ a+2d,\ a+3d, \ldots$$

where a is the first term and d is the common difference.

The nth term is given by $a+(n-1)d$.

2 Geometric Sequences

A geometric sequence has the form

$$a,\ ar,\ ar^2,\ ar^3, \ldots$$

where a is the first term and r is the common ratio.

The nth term is given by ar^{n-1}.

3 Arithmetic Series

The sum to n terms is given by $S_n = \tfrac{n}{2}\left(2a+(n-1)d\right)$.

4 Geometric Series

The sum to n terms is given by $S_n = \dfrac{a\left(1-r^n\right)}{1-r}$.

Provided $-1<r<1$, the sum to infinity is given by $S_\infty = \dfrac{a}{1-r}$.

5 Maclaurin Series

The method is to repeatedly differentiate the function and to substitute $x=0$ into each answer.

The Maclaurin series is then given by

$$\sum_{r=0}^{\infty} f^{(r)}(0)\frac{x^r}{r!} = f(0)+f'(0)\frac{x}{1!}+f''(0)\frac{x^2}{2!}+f^{(3)}(0)\frac{x^3}{3!}+f^{(4)}(0)\frac{x^4}{4!}+\cdots$$

Standard Maclaurin Series

- $e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

These can be used to find other Maclaurin series.

EXAMPLE

$$\begin{aligned} e^{x^2} &= 1 + (x^2) + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + \cdots \\ &= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots \end{aligned}$$

Iterative Schemes

1 Recurrence Relations

A recurrence relation $u_{n+1} = au_n + b$ converges to a limit if $-1 < a < 1$.

The limit is given by $l = \dfrac{b}{1-a}$.

Fixed point

For a recurrence relation $u_{n+1} = au_n + b$ the corresponding equation is $x = ax + b$. The solution of the equation is the same as the limit and is called the **fixed point**.

Calculating roots

To calculate a root using an iterative scheme you must decide whether the scheme converges or diverges.

Convergence is decided by differentiating and substituting in the approximate root.

If the answer is between –1 and 1, the scheme will converge. The closer the value is to zero the quicker the scheme will converge.

Complex Numbers

1 Introduction

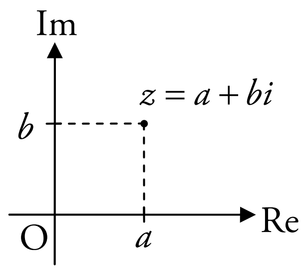
Definition

A complex number hsa the form $z = a + bi$ where a and b are real numbers and $i = \sqrt{-1}$.

a is called the **real part** ($\operatorname{Re} z = a$) and b is the **imaginary part** ($\operatorname{Im} z = b$).

Argand diagram

A complex number $z = a + bi$ can be plotted on an Argand diagram. This is like a Cartesian coordinate diagram but with real (Re) and imaginary (Im) axes.



2 Algebraic Operations

Adding and Subtracting

Adding (or subtracting) complex numbers is done by adding (or subtracting) the real parts and the imaginary parts.

Multiplying

Multiplying two complex numbers is achieved by expanding brackets.

Multiplying by i

Multiplying a complex number by i will rotate it $90°$ anticlockwise about the origin on an Argand diagram.

Complex Conjugate

The complex conjugate of z is denoted $\overline{z}$. If $z = a + bi$ then $\overline{z} = a - bi$.

Geometrically this has the effect of reflecting z in the Re-axis.

Division of complex numbers

Division of two complex numbers can be done using the following process.

1. Find the complex conjugate of the denominator.
2. Multiply top and bottom by the conjugate of the denominator.
3. Express answer in the form $a+bi$.

EXAMPLE

$$\frac{2}{1+i}=\frac{2(1-i)}{(1+i)(1-i)}=\frac{2-2i}{1-i^2}=\frac{2-2i}{2}=1-i.$$

3 Polar Form

The polar form of a complex number $z=a+bi$ is expressed as

$$z=r(\cos\theta+i\sin\theta)$$

where the **modulus** $r=\sqrt{a^2+b^2}$ and the **argument** θ is the angle between the positive Re-axis and the line from the origin to z.

Principle value of the argument

The **principle value** of the argument is the value which lies between $-\pi$ and π.

4 Geometric Interpretations

- $|z|=r$ represents a circle with centre $(0,0)$ and radius r.
- $|z-a|=r$ represents a circle with centre a (in the complex plane) and radius r.
- $|z-a|>r$ represents the points outside the circle stated above.
- $|z-a|<r$ represents the points inside the circle stated above.
- $|z-a|=|z-b|$ represents the points which lie on the perpendicular bisector of the line joining a and b.

5 Fundamental Theorem of Algebra

If a polynomial $P(z)$ has degree n and $P(z)=0$ then there are n solutions (e.g. a quadratic has two solutions, and a cubic has three solutions in $\mathbb{C}$).

- To factorise a quadratic, use the quadratic formula or complete the square.
- To factorise a cubic: find one root by inspection, use algebraic long division to find the quadratic factor. Factorise the quadratic factor (if possible) using the quadratic formula or completing the square.

Note that if z is a root then $\overline{z}$ is also a root.

Multiplication and division in polar form

$$z_1 z_2 = r_1 r_2 \left[\cos\left(\theta_1+\theta_2\right)+i\sin\left(\theta_1+\theta_2\right)\right]$$

When multiplying two complex numbers:
- multiply the moduli;
- add the arguments.

Similarly when dividing two complex numbers:
- divide the moduli;
- subtract the arguments.

6 De Moivre's Theorem

$$z=r\left(\cos\theta+i\sin\theta\right) \quad \Rightarrow \quad z^n=r^n\left(\cos n\theta+i\sin n\theta\right).$$

De Moivre's theorem for fractional powers

$$\left(r\left(\cos\theta+i\sin\theta\right)\right)^{\frac{p}{q}}=r^{\frac{p}{q}}\left(\cos\left(\tfrac{p}{q}\theta\right)+i\sin\left(\tfrac{p}{q}\theta\right)\right).$$

Strategy for finding the cube roots of a complex number

To find the cube roots follow this strategy:

1. Write the number in polar form: $z=r\left(\cos\theta+i\sin\theta\right)$.
2. Write z in two more equivalent ways by adding 2π to the argument.
3. Write down the cube roots by taking the cube root of z (i.e. $z^{\frac{1}{3}}$).

Methods of Proof

Disproving a statement by giving a counter example

If you can find one example that will disprove the statement, then it must be false.

Proof by exhaustion

If the statement is given for a small set of values, then showing that the statement is true for every value will be enough to prove it.

Proof by contradiction

This method assumes that the statement is false and then shows that this leads to something we know to be false (a contradiction). Therefore the original assumption must be false, which means that the statement is true.

Proof by induction

This is a two step process.

1. Show that the statement holds for some value of n (usually $n = 1$).
2. Assume that the statement holds for some constant $n = k$ and *using this assumption* show that the statement holds for $n = k + 1$.

Then the statement holds for all $n \geq 1$.

Direct Proofs

No set rules – just make logical deductions.

Functions

1 Critical and Stationary Points

A **critical point** exists where the derivative is zero or undefined.

Stationary points can be:

- maximum turning points;
- minimum turning points;
- horizontal points of inflection.

2 Derivative Tests

First derivative test

The first derivative test is the nature table method used in Higher.

Second derivative test

If a stationary point exists at $x = a$ then the second derivative test is to calculate $f''(a)$.

- If $f''(a) < 0$ then the point is a maximum turning point.
- If $f''(a) > 0$ then the point is a minimum turning point.

The second derivative test may not always work; if $f''(a) = 0$ then you should revert to the first derivative test.

Non-horizontal points of inflection

A **non-horizontal point of inflection** exists at $x = a$ when $f''(a) = 0$ but $f'(a) \neq 0$ and $f''(x)$ changes sign at $x = a$, i.e. the curve changes from being concave up to concave down or vice versa.

3 Curve Sketching

1. Identify x-axis and y-axis intercepts.
2. Identify turning points and their nature.
3. Consider the behaviour as $x \to \pm\infty$.
   This could be a horizontal asymptote or a slant asymptote (the equation of which is the polynomial part after algebraic long division).
4. Check where y is undefined (this means a vertical asymptote exists).
5. Sketch the curve showing asymptotes, turning points and x- and y-axis intercepts.

Transformations of curves

All transformations from Higher should be known.

The modulus function

To sketch the modulus of a function, take any part of the curve which is below the x-axis and reflect it in the x-axis.

Vectors

1 Direction Ratios and Cosines

Direction Ratios

If a vector is $\boldsymbol{p} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$, then the direction ratio is $a : b : c$.

Direction Cosines

If a vector is $\boldsymbol{p} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ then the direction cosines are $\frac{a}{|\boldsymbol{p}|}$, $\frac{b}{|\boldsymbol{p}|}$ and $\frac{c}{|\boldsymbol{p}|}$.

2 Scalar and Vector Product

Scalar product

The **scalar product** of $\boldsymbol{p} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ and $\boldsymbol{q} = d\boldsymbol{i} + e\boldsymbol{j} + f\boldsymbol{k}$ is

$$\boldsymbol{p}.\boldsymbol{q} = ad + be + cf.$$

Properties

- $\boldsymbol{p}.\boldsymbol{q} = 0 \quad \Leftrightarrow \quad \boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular.
- $\boldsymbol{p}.(\boldsymbol{q} + \boldsymbol{r}) = \boldsymbol{p}.\boldsymbol{q} + \boldsymbol{p}.\boldsymbol{r}$.

Vector product

The **vector product** of $\boldsymbol{p} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ and $\boldsymbol{q} = d\boldsymbol{i} + e\boldsymbol{j} + f\boldsymbol{k}$ is

$$\boldsymbol{p} \times \boldsymbol{q} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a & b & c \\ d & e & f \end{vmatrix} = (bf - ec)\boldsymbol{i} + (af - dc)\boldsymbol{j} + (ae - db)\boldsymbol{k}.$$

The vector product is only defined for vectors in three dimensions.

Geometrically

The magnitude of the vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta \text{ where } \theta \text{ is the angle between } \boldsymbol{a} \text{ and } \boldsymbol{b}.$$

The vector product $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$; its direction is determined by the right hand rule.

Scalar triple product

The scalar triple product is $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c})$. Note that this is a scalar.

3 Equations of Lines

Vector form of the equation of a line

The line through point A with direction $\boldsymbol{d}$ has equation

$$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$$

where $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and λ is a real parameter.

Parametric form of the equation of a line

From the vector equation,

$$x = a_1 + \lambda d_1, \; y = a_2 + \lambda d_2, \; z = a_3 + \lambda d_3$$

where A is the point (a_1, a_2, a_3) and the direction is $\boldsymbol{d} = d_1\boldsymbol{i} + d_2\boldsymbol{j} + d_3\boldsymbol{k}$.

Symmetric form of the equation of a line

If $x = a_1 + \lambda d_1$, $y = a_2 + \lambda d_2$, $z = a_3 + \lambda d_3$ are the parametric equations of a line, then the symmetric equation is

$$\frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}$$

This is obtained by rearranging the parametric equations for λ.

4 Equations of Planes

Vector equation of a plane

The vector equation of the plane containing point A with normal $\boldsymbol{n}$ is

$$\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{a}.\boldsymbol{n}$$

where $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

To find the vector equation of the plane containing the non-collinear points A, B and C, use the following process.

1. Calculate $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and call one of these $\boldsymbol{a}$.
2. Calculate the normal as $\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC}$.
3. Using $\boldsymbol{n}$ and $\boldsymbol{a}$ write down $\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{a}.\boldsymbol{n}$.

Cartesian equation of a plane

The Cartesian equation of a plane containing the point $A(a_1, a_2, a_3)$, with normal $\boldsymbol{n} = n_1\boldsymbol{i} + n_2\boldsymbol{j} + n_3\boldsymbol{k}$ is

$$n_1x + n_2y + n_3z = d$$

where $d = \boldsymbol{a}.\boldsymbol{n}$. (This comes from the vector equation.)

Parametric equation of a plane

The parametric equation of a plane is

$$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$$

where $\boldsymbol{a}$ is the position vector of a point in the plane, $\boldsymbol{b}$ and $\boldsymbol{c}$ are two non-collinear vectors parallel to the plane, and λ and μ are real parameters.

5 Intersections

Intersection of, and the angle between two lines

Two lines in three dimensions can

- be identical;
- be parallel;
- intersect;
- be skew (not intersect at all).

The angle between two lines L_1 and L_2 is calculated by finding their directions $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$, and then using the rearranged form of the scalar product formula from Higher:

$$\cos\theta = \frac{\boldsymbol{d}_1.\boldsymbol{d}_2}{|\boldsymbol{d}_1||\boldsymbol{d}_2|}.$$

Intersection of, and angle between two planes

Two planes in three dimensions can

- be coincident, i.e. the same plane;
- be parallel;
- intersect in a line.

Note that parallel and coincident planes have normals that are proportional.

The intersection of two planes can be found using Gaussian elimination or algebraic manipulation to solve their equations simultaneously.

The angle between two planes is the angle between their normals. Given planes with normals $\boldsymbol{m}$ and $\boldsymbol{n}$, the angle between them is calculated using:

$$\cos\theta = \frac{\boldsymbol{n}.\boldsymbol{m}}{|\boldsymbol{n}||\boldsymbol{m}|}.$$

Intersection of three planes

Three planes in three dimensions can

- intersect at a point;
- intersect in a line;
- have no common point of intersection.

Use Gaussian elimination to work out how the planes intersect:

- a unique solution means intersection at a point;
- a family of solutions means intersection in a line;
- no solution means no common point of intersection.

Intersection of, and the angle between a line and a plane

A point of intersection can found by substituting the equation of the line in parametric form into the equation of the plane and solving for λ.

To find the angle between a line and a plane:

1. find $\boldsymbol{n}$, the normal to the plane;
2. find the angle θ between the line and $\boldsymbol{n}$;
3. the angle between the line and the plane is $\frac{\pi}{2} - \theta$.

Differentiation

1 Rules

Product Rule

$$(fg)' = f'g + fg'.$$

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

Chain Rule

$$\frac{d}{dx}\big(f(g(x))\big) = f'(g(x)).g'(x).$$

Trigonometric Definitions

$$\sec x = \frac{1}{\cos x} \qquad \operatorname{cosec} x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x}$$

2 Standard Derivatives

$$\frac{d}{dx}\left(e^x\right) = e^x \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\sin^{-1} \tfrac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} \tfrac{x}{a}\right) = \frac{-1}{\sqrt{a^2-x^2}} \qquad \frac{d}{dx}\left(\tan^{-1} \tfrac{x}{a}\right) = \frac{1}{a^2+x^2}$$

3 Inverse Differentiation

- If $f(x)$ has inverse $f^{-1}(x)$ then $\dfrac{d}{dx} f^{-1}(x) = \dfrac{1}{f'\left(f^{-1}(x)\right)}$.
- $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$.

4 Implicit Differentiation

This is done using the chain rule: $\frac{d}{dx}f(y)=\frac{d}{dy}f(y).\frac{dy}{dx}$.

EXAMPLE

Given $x^2+xy+y^2=1$, find $\frac{dy}{dx}$.

Differentiating throughout with respect to x:

$$\begin{aligned}\tfrac{d}{dx}\left(x^2\right)+\tfrac{d}{dx}(xy)+\tfrac{d}{dy}\left(y^2\right)\tfrac{dy}{dx}&=\tfrac{d}{dx}(1)\\ 2x+y+x\frac{dy}{dx}+2y\frac{dy}{dx}&=0\\ (x+2y)\frac{dy}{dx}&=-2x-y\\ \frac{dy}{dx}&=-\frac{2x+y}{x+2y}.\end{aligned}$$

5 Parametric Differentiation

If $x=f(t)$ and $y=f(t)$ then $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

EXAMPLE

A curve is defined parametrcally by $x=2t+1$, $y=2t^3$.

Find $\frac{dy}{dx}$ in terms of t.

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{6t^2}{2}=3t^2.$$

Integration

1 Standard Integrals

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad \int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

$$\int \sin x \, dx = -\cos x + c \qquad \int \cos x \, dx = \sin x + c$$

$$\int \sin(ax+b) \, dx = -\tfrac{1}{a}\cos(ax+b) + c \qquad \int \cos(ax+b) \, dx = \tfrac{1}{a}\sin(ax+b) + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + c \qquad \int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\tfrac{x}{a}\right) + c$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c \qquad \int \frac{1}{a^2-x^2} \, dx = \tfrac{1}{a}\tan^{-1}\left(\tfrac{x}{a}\right) + c$$

$$\int e^x \, dx = e^x + c \qquad \int e^{ax+b} \, dx = \tfrac{1}{a} e^{ax+b} + c$$

$$\int \tfrac{1}{x} \, dx = \ln|x| + c \qquad \int \frac{1}{ax+b} \, dx = \tfrac{1}{a}\ln|ax+b| + c$$

2 Integration by Substitution

To find $\int f(x) \, dx$ using the substitution $u = u(x)$

1. find $\dfrac{du}{dx}$ and rearrange to get dx in terms of du;
2. replace dx by this expression and replace $u(x)$ with u;
3. integrate with respect to u.

EXAMPLE

Use the substitution $u = x^2 - 8$ to find $\int x^3 \left(x^2 - 8\right)^{\frac{1}{3}} dx$.

Given $u = x^2 - 8$, $\dfrac{du}{dx} = 2x$ so $dx = \dfrac{du}{2x}$. Then

$$\begin{aligned}
\int x^3 \left(x^2 - 8\right)^{\frac{1}{3}} dx &= \int x^3 u^{\frac{1}{3}} \frac{du}{2x} \\
&= \tfrac{1}{2} \int x^2 u^{\frac{1}{3}} du \\
&= \tfrac{1}{2} \int (u + 8) u^{\frac{1}{3}} du \\
&= \tfrac{1}{2} \int \left(u^{\frac{4}{3}} + 8u^{\frac{1}{3}}\right) du \\
&= \tfrac{1}{2} \left(\tfrac{3}{7} u^{\frac{7}{3}} + \tfrac{24}{4} u^{\frac{4}{3}}\right) + c \\
&= \tfrac{3}{14} \left(x^2 - 8\right)^{\frac{7}{3}} + 3\left(x^2 - 8\right)^{\frac{4}{3}} + c.
\end{aligned}$$

To find a definite integral, you should change the limits to be in terms of u.

3 Areas

The area between two curves or between a curve and an axis can be calculated as in Higher.

4 Volumes of Solids of Revolution

The solid formed by rotating a curve between $x = a$ and $x = b$ about the x-axis has volume $V = \int_a^b \pi y^2 \, dx$.

5 Partial fractions and Integration

Partial fractions can be used to integration rational functions.

EXAMPLE

$$\int \frac{1}{(x+1)(x-3)} dx = \tfrac{1}{4} \int \left(\frac{1}{x-3} - \frac{1}{x+1}\right) dx = \tfrac{1}{4}\left(\ln|x-3| - \ln|x+1|\right) + c.$$

6 Integration by Parts

$$\int fg' = fg - \int f'g$$

Sometimes one application of integration by parts is not enough and a repeated application must be used.

Differential Equations

1 Separable Differential Equations

If $\frac{dy}{dx} = f(x)g(y)$ then $\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$.

A general solution has a "$+c$" and a particular solution can be found if initial conditions are given.

2 First Order Linear Differential Equations

These have the form

$$\frac{dy}{dx} + P(x)y = f(x)$$

and can be solved using the following process.

1. Identify $P(x)$ (the equation must be in the form above).
2. Calculate the integrating factor $I(x) = e^{\int P(x)\,dx}$.
3. Then $\frac{d}{dx}\big(I(x)y\big) = I(x)f(x)$.
4. Integrate both sides to get $I(x)y = \int \big(I(x)f(x)\big)dx$.
5. Rearrange to give y.

If given initial conditions then a particular solution can be found.

3 Second Order Homogeneous Linear Differential Equations

These have the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

and can be solved using the following process.

1. Write down auxiliary equation: $am^2 + bm + c = 0$.
2. Solve for m and write down the general solution as follows
   - distinct real roots m_1 and m_2: $y = Ae^{m_1x} + Be^{m_2x}$;
   - one repeated real root m: $y = Ae^{mx} + Bxe^{mx}$;
   - complex roots $p \pm qi$: $y = e^{px}\left(A\cos qx + B\sin qx\right)$.

4 Second Order Non-Homogeneous Differential Equations

These have the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

where $f(x)$ will be a simple function such as a polynomial, exponential or trigonometric function.

The particular integral y_p is calculated by making an educated guess:

- if $f(x)$ is a constant, try $y_p=a$
- if $f(x)$ is linear, try $y_p=ax+b$
- if $f(x)$ is a quadratic, try $y_p=ax^2+bx+c$
- if $f(x)=e^{rx}$, try $y_p=ke^{rx}$
- if $f(x)$ has the form $\sin nx$ or $\cos nx$, try $y_p=p\sin nx+q\cos nx$.

The particular integral is then found by substituting y_p, $\frac{dy_p}{dx}$ and $\frac{d^2y_p}{dx^2}$ into the differential equation and solving by equating coefficients.

The general solution is given by

$$y=y_c+y_p$$

where y_c is the complementary solution, i.e. the solution of the corresponding homogeneous equation:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$