



Higher Mathematics

UNIT 3

Mathematics 3

HSN23000

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Contents

Vectors	128
1 Vectors and Scalars	128
2 Components	128
3 Magnitude	130
4 Equal Vectors	131
5 Addition and Subtraction of Vectors	132
6 Multiplication by a Scalar	134
7 Position Vectors	135
8 Basis Vectors	136
9 Collinearity	137
10 Dividing Lines in a Ratio	138
11 The Scalar Product	141
12 The Angle Between Vectors	144
13 Perpendicular Vectors	147
14 Properties of the Scalar Product	148
Further Calculus	149
1 Differentiating $\sin x$ and $\cos x$	149
2 Integrating $\sin x$ and $\cos x$	150
3 The Chain Rule	151
4 Special Cases of the Chain Rule	151
5 A Special Integral	154
6 Integrating $\sin(ax + b)$ and $\cos(ax + b)$	157
Exponentials and Logarithms	160
1 Exponentials	160
2 Logarithms	162
3 Laws of Logarithms	162
4 Exponentials and Logarithms to the Base e	165
5 Exponential and Logarithmic Equations	166
6 Graphing with Logarithmic Axes	169
7 Graph Transformations	173
Wave Functions	175
1 Expressing $p\cos x + q\sin x$ in the form $k\cos(x - a)$	175
2 Expressing $p\cos x + q\sin x$ in other forms	176
3 Multiple Angles	177
4 Maximum and Minimum Values	178
5 Solving Equations	179
6 Sketching Graphs of $y = p\cos x + q\sin x$	181

OUTCOME 1

Vectors

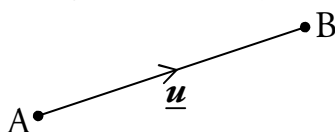
1 Vectors and Scalars

A **scalar** is a quantity with magnitude (size) only – for example, an amount of money or a length of time.

Sometimes size alone is not enough to describe a quantity – for example, directions to the nearest shop. For this we need to know a magnitude (i.e. how far), *and* a direction.

Quantities with both magnitude and direction are called **vectors**.

A vector is named either by using the points at the end of a directed line segment (e.g. \overrightarrow{AB} represents the vector starting at point A and ending at point B) or by using a bold letter (e.g. \mathbf{u}). You will see bold letters used in printed text, but in handwriting you should just underline the letter (e.g. \underline{u}).



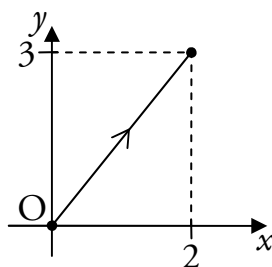
Throughout these notes, we will show vectors bold and underlined (e.g. \underline{u}).

2 Components

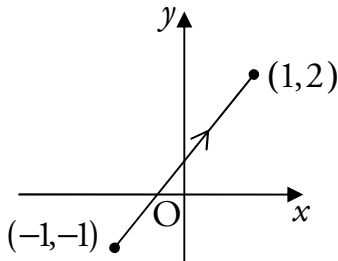
A vector may be represented by its **components**, which we write in a column. For example,

$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is a vector in two dimensions.

In this case the first component is 2 and this tells us to move 2 units in the x -direction. The second component tells us to move 3 units in the y -direction. So if the vector starts at the origin, it will look like:



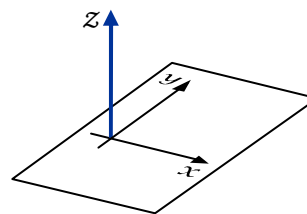
Note that we write the components in a column to avoid confusing them with coordinates. The following diagram also shows the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, but in this case it does not start at the origin.



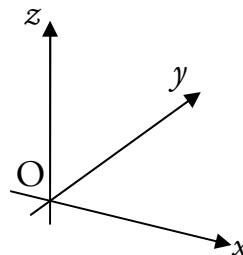
Vectors in Three Dimensions

In a vector with three components, the first two tell us how many units to move in the x - and y -directions, as before. The third component specifies how far to move in the z -direction.

When looking at a pair of (x, y) -axes, the z -axis points out of the page from the origin.



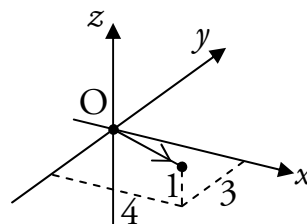
A set of 3D axes can be drawn on a page as shown to the right.



For example,

$$\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

is a vector in three dimensions. This vector is shown in the diagram, starting from the origin.



Zero Vectors

Any vector with all its components zero is called a zero vector and can be

written as $\underline{\mathbf{0}}$, e.g. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \underline{\mathbf{0}}$.

3 Magnitude

The **magnitude** (or length) of a vector \underline{u} is written as $|\underline{u}|$. It can be calculated as follows.

$$\text{If } \overrightarrow{PQ} = \begin{pmatrix} a \\ b \end{pmatrix} \text{ then } |\overrightarrow{PQ}| = \sqrt{a^2 + b^2}.$$

$$\text{If } \overrightarrow{PQ} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ then } |\overrightarrow{PQ}| = \sqrt{a^2 + b^2 + c^2}.$$

EXAMPLES

1. Given $\underline{u} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$, find $|\underline{u}|$.

$$\begin{aligned} |\underline{u}| &= \sqrt{5^2 + (-12)^2} \\ &= \sqrt{169} \\ &= 13 \text{ units.} \end{aligned}$$

2. Find the length of $\underline{a} = \begin{pmatrix} -\sqrt{5} \\ 6 \\ 3 \end{pmatrix}$.

$$\begin{aligned} |\underline{a}| &= \sqrt{(-\sqrt{5})^2 + 6^2 + 3^2} \\ &= \sqrt{50} \\ &= 5\sqrt{2} \text{ units.} \end{aligned}$$

Unit Vectors

Any vector with a magnitude of one is called a **unit vector**. For example:

$$\begin{aligned} \text{if } \underline{u} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix} \text{ then } |\underline{u}| &= \sqrt{\left(\frac{1}{2}\right)^2 + 0^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \sqrt{\frac{4}{4}} \\ &= 1 \text{ unit.} \end{aligned}$$

So \underline{u} is a unit vector.

Distance in Three Dimensions

The distance between the points A and B is $d_{AB} = |\overline{AB}|$ units.

So given $\overline{AB} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$, we find $d_{AB} = \sqrt{(-1)^2 + 2^2 + 5^2} = \sqrt{30}$.

In fact, there is a three dimensional version of the distance formula.

The distance d between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \text{ units.}$$

EXAMPLE

Find the distance between the points $(-1, 4, 1)$ and $(0, 5, -7)$.

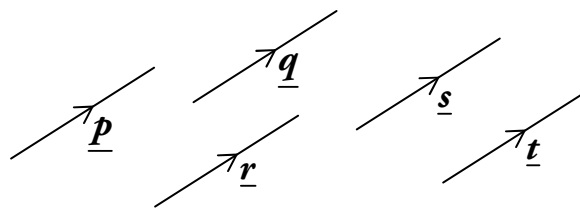
$$\begin{aligned} \text{The distance is } & \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ & = \sqrt{(0 - (-1))^2 + (5 - 4)^2 + (-7 - 1)^2} \\ & = \sqrt{1^2 + 1^2 + (-8)^2} \\ & = \sqrt{1 + 1 + 64} \\ & = \sqrt{66} \text{ units.} \end{aligned}$$

4 Equal Vectors

Vectors with the same magnitude and direction are said to be equal.

For example, all the vectors shown to the right are equal.

If vectors are equal to each other, then all of their components are equal, i.e.

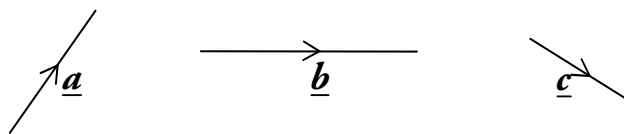


$$\text{if } \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} d \\ e \\ f \end{pmatrix} \text{ then } a = d, b = e \text{ and } c = f.$$

Conversely, two vectors are only equal if all of their components are equal.

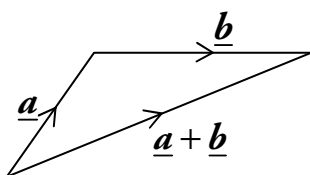
5 Addition and Subtraction of Vectors

Consider the following vectors:



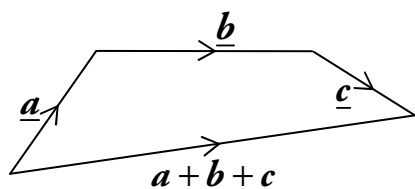
Addition

We can construct $\underline{a} + \underline{b}$ as follows:



$\underline{a} + \underline{b}$ means \underline{a} followed by \underline{b} .

Similarly, we can construct $\underline{a} + \underline{b} + \underline{c}$ as follows:

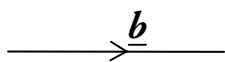


$\underline{a} + \underline{b} + \underline{c}$ means \underline{a} followed by \underline{b} followed by \underline{c} .

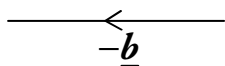
To add vectors, we position them nose-to-tail. Then the sum of the vectors is the vector between the first tail and the last nose.

Subtraction

Now consider $\underline{a} - \underline{b}$. This can be written as $\underline{a} + (-\underline{b})$, so if we first find $-\underline{b}$ we can use vector addition to obtain $\underline{a} - \underline{b}$.

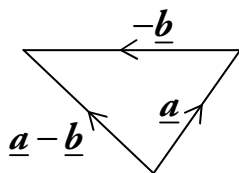


$-\underline{b}$ is just \underline{b} but in the opposite direction.



$-\underline{b}$ and \underline{b} have the same magnitude, i.e. $|\underline{b}| = |-\underline{b}|$.

Therefore we can construct $\underline{a} - \underline{b}$ as follows:



$\underline{a} - \underline{b}$ means \underline{a} followed by $-\underline{b}$.

Using Components

If we have the components of vectors, then things become much simpler.

The following rules can be used for addition and subtraction.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix}$$

add the components

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a-d \\ b-e \\ c-f \end{pmatrix}$$

subtract the components

EXAMPLES

1. Given $\underline{u} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, calculate $\underline{u} + \underline{v}$ and $\underline{u} - \underline{v}$.

$$\underline{u} + \underline{v} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 7 \\ 2 \end{pmatrix}$$

$$\underline{u} - \underline{v} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

2. Given $\underline{p} = \begin{pmatrix} 4 \\ \frac{3}{2} \\ 3 \end{pmatrix}$ and $\underline{q} = \begin{pmatrix} -1 \\ 3 \\ -\frac{6}{5} \end{pmatrix}$, calculate $\underline{p} - \underline{q}$ and $\underline{q} + \underline{p}$.

$$\underline{p} - \underline{q} = \begin{pmatrix} 4 \\ \frac{3}{2} \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ -\frac{6}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ -\frac{3}{2} \\ \frac{21}{5} \end{pmatrix}$$

$$\underline{q} + \underline{p} = \begin{pmatrix} -1 \\ 3 \\ -\frac{6}{5} \end{pmatrix} + \begin{pmatrix} 4 \\ \frac{3}{2} \\ 3 \end{pmatrix}$$

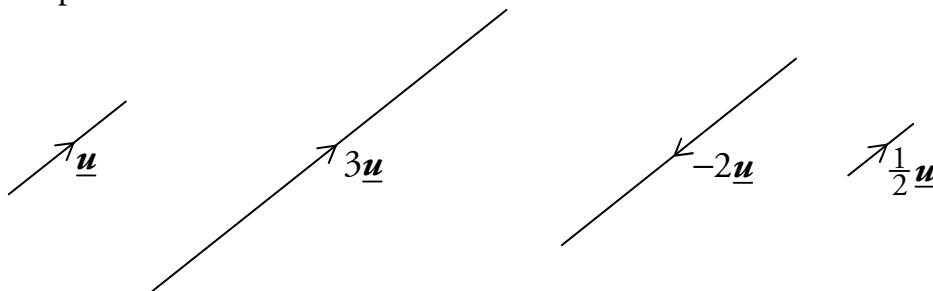
$$= \begin{pmatrix} 3 \\ \frac{9}{2} \\ \frac{9}{5} \end{pmatrix}.$$

6 Multiplication by a Scalar

A vector \underline{u} which is multiplied by a scalar $k > 0$ will give the result $k\underline{u}$. This vector will be k times as long, i.e. its magnitude will be $k|\underline{u}|$.

Note that if $k < 0$ this means that the vector $k\underline{u}$ will be in the opposite direction to \underline{u} .

For example:



$$\text{If } \underline{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ then } k\underline{u} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}.$$

Each component is multiplied by the scalar.

EXAMPLES

1. Given $\underline{v} = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$, find $3\underline{v}$.

$$3\underline{v} = 3 \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 15 \\ -9 \end{pmatrix}.$$

2. Given $\underline{r} = \begin{pmatrix} -6 \\ 3 \\ 1 \end{pmatrix}$, find $-4\underline{r}$.

$$-4\underline{r} = -4 \begin{pmatrix} -6 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 24 \\ -12 \\ -4 \end{pmatrix}.$$

Negative Vectors

The negative of a vector is the vector multiplied by -1 .

If we write a vector as a directed line segment \overline{AB} , then $-\overline{AB} = \overline{BA}$:

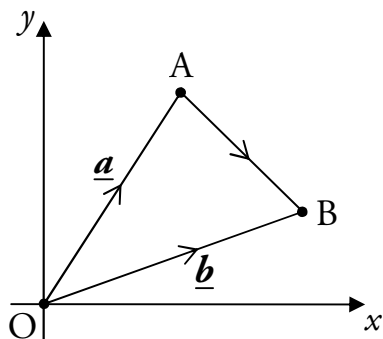
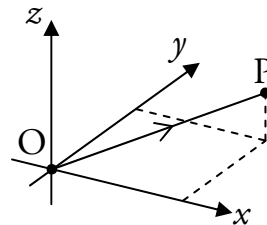


7 Position Vectors

\overline{OA} is called the **position vector** of point A relative to the origin O, and is written as \underline{a} .

\overline{OB} is called the position vector of point B, written \underline{b} .

Given $P(x, y, z)$, the position vector \overline{OP} or \underline{p} has components $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.



To move from point A to point B we can move back along the vector \underline{a} to the origin, and along vector \underline{b} to point B, i.e.

$$\begin{aligned} \overline{AB} &= \overline{AO} + \overline{OB} \\ &= -\overline{OA} + \overline{OB} \\ &= -\underline{a} + \underline{b} \\ &= \underline{b} - \underline{a}. \end{aligned}$$

For the vector joining any two points P and Q, $\overline{PQ} = \underline{q} - \underline{p}$.

EXAMPLE

R is the point $(2, -2, 3)$ and S is the point $(4, 6, -1)$. Find \overline{RS} .

From the coordinates, $\underline{r} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ and $\underline{s} = \begin{pmatrix} 4 \\ 6 \\ -1 \end{pmatrix}$.

$$\begin{aligned} \overline{RS} &= \underline{s} - \underline{r} \\ &= \begin{pmatrix} 4 \\ 6 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix}. \end{aligned}$$

Note

You don't need to write this line down in the exam.

8 Basis Vectors

A vector may also be defined in terms of the basis vectors \underline{i} , \underline{j} and \underline{k} .

These are three mutually perpendicular unit vectors (i.e. they are perpendicular to each other).



These basis vectors can be written in component form as

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Any vector can be written in **basis form** using \underline{i} , \underline{j} and \underline{k} . For example:

$$\underline{v} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2\underline{i} - 3\underline{j} + 6\underline{k}.$$

There is no need for the working above if the following is used:

$$a\underline{i} + b\underline{j} + c\underline{k} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

9 Collinearity

In Straight Lines (Unit 1 Outcome 1), we learned that points are collinear if they lie on the same straight line.

The points A, B and C in 3D space are collinear if \overline{AB} is parallel to \overline{BC} , with B a common point.

Note that we *cannot* find gradients in three dimensions – instead we use the following.

Non-zero vectors are **parallel** if they are scalar multiples of the same vector.

For example:

$$\underline{u} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \quad \underline{v} = \begin{pmatrix} 6 \\ 3 \\ 12 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 3\underline{u}.$$

So \underline{u} and \underline{v} are parallel.

$$\underline{p} = \begin{pmatrix} 15 \\ 9 \\ -6 \end{pmatrix} = 3 \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}, \quad \underline{q} = \begin{pmatrix} 20 \\ 12 \\ -8 \end{pmatrix} = 4 \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}.$$

So \underline{p} and \underline{q} are parallel.

EXAMPLE

A is the point (1, -2, 5), B(8, -5, 9) and C(22, -11, 17).

Show that A, B and C are collinear.

$$\overline{AB} = \underline{b} - \underline{a}$$

$$= \begin{pmatrix} 8 \\ -5 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ -3 \\ 4 \end{pmatrix}$$

$$\overline{BC} = \underline{c} - \underline{b}$$

$$= \begin{pmatrix} 22 \\ -11 \\ 17 \end{pmatrix} - \begin{pmatrix} 8 \\ -5 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} 14 \\ -6 \\ 8 \end{pmatrix} = 2 \begin{pmatrix} 7 \\ -3 \\ 4 \end{pmatrix}.$$

$\overline{BC} = 2\overline{AB}$, so \overline{AB} and \overline{BC} are parallel, and since B is a common point, A, B and C are collinear.

10 Dividing Lines in a Ratio

There is a simple process for finding the coordinates of a point which divides a line segment in a given ratio.

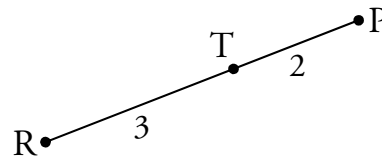
EXAMPLE

1. P is the point $(-2, 4, -1)$ and R is the point $(8, -1, 19)$.

The point T divides PR in the ratio $2 : 3$. Find the coordinates of T.

Step 1

Make a sketch of the line, showing the ratio in which the point divides the line segment.



Step 2

Using the sketch, equate the ratio of the two line segments with the given ratio.

$$\frac{PT}{TR} = \frac{2}{3}$$

Step 3

Cross multiply, then change directed line segments to position vectors.

$$3\overline{PT} = 2\overline{TR}$$

$$3(\underline{t} - \underline{p}) = 2(\underline{r} - \underline{t})$$

Step 4

Rearrange to give the position vector of the unknown point.

$$3\underline{t} - 3\underline{p} = 2\underline{r} - 2\underline{t}$$

$$3\underline{t} + 2\underline{t} = 2\underline{r} + 3\underline{p}$$

$$5\underline{t} = 2 \begin{pmatrix} 8 \\ -1 \\ 19 \end{pmatrix} + 3 \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}$$

$$5\underline{t} = \begin{pmatrix} 16 \\ -2 \\ 38 \end{pmatrix} + \begin{pmatrix} -6 \\ 12 \\ -3 \end{pmatrix}$$

$$5\underline{t} = \begin{pmatrix} 10 \\ 10 \\ 35 \end{pmatrix}$$

$$\underline{t} = \begin{pmatrix} 2 \\ 2 \\ 7 \end{pmatrix}$$

Step 5

From the position vector, state the coordinates of the unknown point.

So T is the point $(2, 2, 7)$.

Using the Section Formula

The previous method can be condensed into a formula as shown below.

If the point P divides the line AB in the ratio $m : n$, then

$$\underline{p} = \frac{n\underline{a} + m\underline{b}}{n + m},$$

where \underline{a} , \underline{b} and \underline{p} are the position vectors of A, B and P respectively.

This is referred to as the **section formula**.

It is not necessary to know this, since the approach explained above will always work.

EXAMPLE

2. P is the point $(-2, 4, -1)$ and R is the point $(8, -1, 19)$.

The point T divides PR in the ratio $2 : 3$. Find the coordinates of T.

The ratio is $2 : 3$, so $m = 2$ and $n = 3$. Hence:

$$\begin{aligned} \underline{t} &= \frac{n\underline{p} + m\underline{r}}{n + m} \\ &= \frac{3\underline{p} + 2\underline{r}}{5} \\ &= \left(\begin{array}{l} \frac{1}{5}(3(-2) + 2(8)) \\ \frac{1}{5}(3(4) + 2(-1)) \\ \frac{1}{5}(3(-1) + 2(19)) \end{array} \right) \\ &= \left(\begin{array}{l} 2 \\ 2 \\ 7 \end{array} \right). \end{aligned}$$

So T is the point $(2, 2, 7)$.

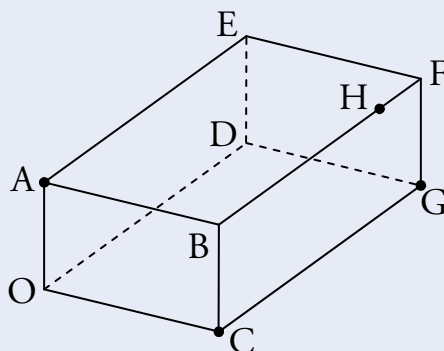
Note

If you are confident with arithmetic, this step can be done mentally.

Further Examples

EXAMPLES

3. The cuboid OABCDEFG is shown in the diagram below.



The point A has coordinates $(0,0,5)$, C $(8,0,0)$ and G $(8,12,0)$. The point H divides BF in the ratio $4:1$. Find the coordinates of H.

From the diagram:

$$\begin{aligned}\overrightarrow{OH} &= \overrightarrow{OA} + \overrightarrow{AB} + \frac{4}{5}\overrightarrow{BF} \\ &= \overrightarrow{OA} + \overrightarrow{OC} + \frac{4}{5}\overrightarrow{CG} \\ \underline{h} &= \underline{a} + \underline{c} + \frac{4}{5}(\underline{g} - \underline{c}) \\ &= \underline{a} + \underline{c} + \frac{4}{5}\underline{g} - \frac{4}{5}\underline{c} \\ &= \underline{a} + \frac{1}{5}\underline{c} + \frac{4}{5}\underline{g} \\ &= \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 8 \\ 0 \\ 0 \end{pmatrix} + \frac{4}{5}\begin{pmatrix} 8 \\ 12 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 8 \\ \frac{48}{5} \\ 5 \end{pmatrix}.\end{aligned}$$

So H has coordinates $(8, \frac{48}{5}, 5)$.

4. The points P $(6,1,-3)$, Q $(8,-3,1)$ and R $(9,-5,3)$ are collinear. Find the ratio in which Q divides PR.

Since the points are collinear $\overrightarrow{PQ} = k\overrightarrow{QR}$ for some k . Working with the first components:

$$\begin{aligned}8 - 6 &= k(9 - 8) \\ k &= 2.\end{aligned}$$

Therefore $\overrightarrow{PQ} = 2\overrightarrow{QR}$ so Q divides PR in the ratio $2:1$.

Note

$$\frac{BQ}{QR} = \frac{2}{1}, \text{ so } \overrightarrow{BQ} = \frac{2}{3}\overrightarrow{BR}.$$

Note

The ratio is $2:1$ since

$$\frac{PQ}{QR} = \frac{2}{1}.$$

5. The points $A(7, -4, -4)$, $B(13, 5, -7)$ and C are collinear. Given that B divides AC in the ratio $3:2$, find the coordinates of C .

$$\overline{AB} = \frac{3}{5} \overline{AC}$$

$$\underline{b} - \underline{a} = \frac{3}{5}(\underline{c} - \underline{a})$$

$$\underline{b} - \underline{a} = \frac{3}{5}\underline{c} - \frac{3}{5}\underline{a}$$

$$\frac{3}{5}\underline{c} = \underline{b} - \frac{2}{5}\underline{a}$$

$$\underline{c} = \frac{5}{3}\underline{b} - \frac{2}{3}\underline{a}$$

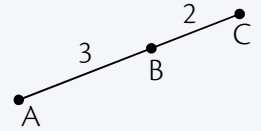
$$= \frac{5}{3} \begin{pmatrix} 13 \\ 5 \\ -7 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 7 \\ -4 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 17 \\ 11 \\ -9 \end{pmatrix}.$$

So C has coordinates $(17, 11, -9)$.

Note

A sketch may help you to see this:



11 The Scalar Product

So far we have added and subtracted vectors and multiplied a vector by a scalar. Now we will consider the scalar product, which is a form of vector multiplication.

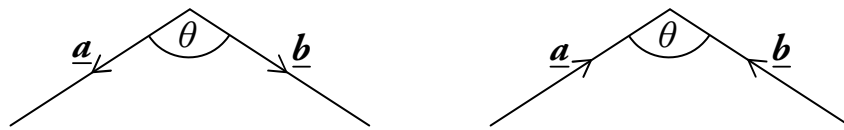
The **scalar product** is denoted by $\underline{a} \cdot \underline{b}$ (sometimes it is called the dot product) and can be calculated using the formula:

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos \theta,$$

where θ is the angle between the two vectors \underline{a} and \underline{b} .

This is given in the exam.

The definition above assumes that the vectors \underline{a} and \underline{b} are positioned so that they both point away from the angle, or both point into the angle.



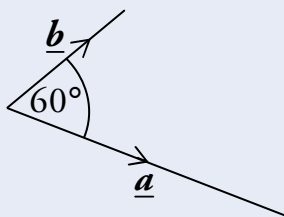
However, if one vector is pointing away from the angle, while the other points into the angle,



we find that $\underline{a} \cdot \underline{b} = -|\underline{a}||\underline{b}|\cos\theta$.

EXAMPLES

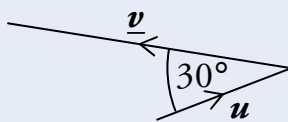
- Two vectors, \underline{a} and \underline{b} have magnitudes 7 and 3 units respectively and are at an angle of 60° to each other as shown below.



What is the value of $\underline{a} \cdot \underline{b}$?

$$\begin{aligned}\underline{a} \cdot \underline{b} &= |\underline{a}||\underline{b}|\cos\theta \\ &= 7 \times 3 \times \cos 60^\circ \\ &= 21 \times \frac{1}{2} \\ &= \frac{21}{2}.\end{aligned}$$

- The vector \underline{u} has magnitude k and \underline{v} is twice as long as \underline{u} . The angle between \underline{u} and \underline{v} is 30° , as shown below.



Find an expression for $\underline{u} \cdot \underline{v}$ in terms of k .

$$\begin{aligned}\underline{u} \cdot \underline{v} &= -|\underline{u}||\underline{v}|\cos\theta \\ &= -k \times 2k \times \cos 30^\circ \\ &= -2k^2 \times \frac{\sqrt{3}}{2} \\ &= -\sqrt{3}k^2.\end{aligned}$$

Remember

When one vector points in and one points out,
 $\underline{u} \cdot \underline{v} = -|\underline{u}||\underline{v}|\cos\theta$.

The Component Form of the Scalar Product

The scalar product can also be calculated as follows:

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad \text{where } \underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

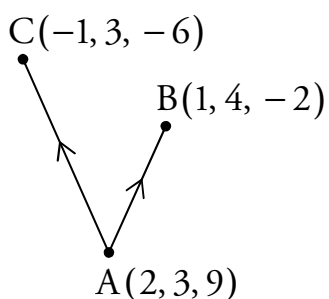
This is given in the exam.

EXAMPLES

3. Find $\underline{p} \cdot \underline{q}$, given that $\underline{p} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\underline{q} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$.

$$\begin{aligned} \underline{p} \cdot \underline{q} &= p_1 q_1 + p_2 q_2 + p_3 q_3 \\ &= (1 \times 2) + (2 \times 2) + ((-3) \times 3) \\ &= 2 + 4 - 9 \\ &= -3. \end{aligned}$$

4. If A is the point (2, 3, 9), B(1, 4, -2) and C(-1, 3, -6), calculate $\overline{AB} \cdot \overline{AC}$.



We need to use the position vectors of the points:

$$\begin{aligned} \overline{AB} &= \underline{b} - \underline{a} & \overline{AC} &= \underline{c} - \underline{a} \\ &= \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix} & &= \begin{pmatrix} -1 \\ 3 \\ -6 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \\ -11 \end{pmatrix} & &= \begin{pmatrix} -3 \\ 0 \\ -15 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} \overline{AB} \cdot \overline{AC} &= ((-1) \times (-3)) + (1 \times 0) + ((-11) \times (-15)) \\ &= 3 + 0 + 165 \\ &= 168. \end{aligned}$$

12 The Angle Between Vectors

The formulae for the scalar product can be rearranged to give the following equations, both of which can be used to calculate θ , the angle between two vectors.

$$\cos \theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}| |\underline{b}|} \quad \text{or} \quad \cos \theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{|\underline{a}| |\underline{b}|}.$$

Look back to the formulae for finding the scalar product, given on the previous pages. Notice that the first equation is simply a rearranged form of the one which can be used to find the scalar product. Also notice that the second simply substitutes $\underline{a} \cdot \underline{b}$ for the component form of the scalar product.

These formulae are *not* given in the exam but can both be easily derived from the formulae on the previous pages (which *are* given in the exam).

EXAMPLES

1. Calculate the angle θ between vectors $\underline{p} = 3\underline{i} + 4\underline{j} - 2\underline{k}$ and $\underline{q} = 4\underline{i} + \underline{j} + 3\underline{k}$.



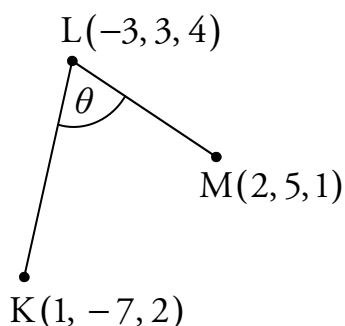
$$\underline{p} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \quad \text{and} \quad \underline{q} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{aligned} \cos \theta &= \frac{p_1 q_1 + p_2 q_2 + p_3 q_3}{|\underline{p}| |\underline{q}|} \\ &= \frac{(3 \times 4) + (4 \times 1) + ((-2) \times 3)}{\sqrt{3^2 + 4^2 + (-2)^2} \sqrt{4^2 + 1^2 + 3^2}} \\ &= \frac{10}{\sqrt{29} \sqrt{26}} \\ \theta &= \cos^{-1} \left(\frac{10}{\sqrt{29} \sqrt{26}} \right) \\ &= 68.6^\circ \text{ (to 1 d.p.)} \quad (\text{or } 1.198 \text{ radians (to 3 d.p.)}) \end{aligned}$$



2. K is the point $(1, -7, 2)$, $L(-3, 3, 4)$ and $M(2, 5, 1)$. Find \widehat{KLM} .

Start with a sketch:



Now find the vectors pointing away from the angle:

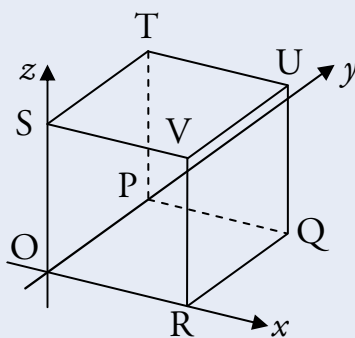
$$\overline{LK} = \underline{k} - \underline{l} = \begin{pmatrix} 1 \\ -7 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -10 \\ -2 \end{pmatrix},$$

$$\overline{LM} = \underline{m} - \underline{l} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}.$$

Use the scalar product to find the angle:

$$\begin{aligned} \cos \widehat{KLM} &= \frac{\overline{LK} \cdot \overline{LM}}{|\overline{LK}| |\overline{LM}|} \\ &= \frac{(4 \times 5) + (-10 \times 2) + (-2 \times (-3))}{\sqrt{4^2 + (-10)^2 + (-2)^2} \sqrt{5^2 + 2^2 + (-3)^2}} \\ &= \frac{6}{\sqrt{120} \sqrt{38}} \\ \widehat{KLM} &= \cos^{-1} \left(\frac{6}{\sqrt{120} \sqrt{38}} \right) \\ &= 84.9^\circ \text{ (to 1 d.p.)} \quad (\text{or } 1.482 \text{ radians (to 3 d.p.)}) \end{aligned}$$

3. The diagram below shows the cube OPQRSTUV.



The point R has coordinates $(4, 0, 0)$.

- (a) Write down the coordinates of T and U.
 (b) Find the components of \overrightarrow{RT} and \overrightarrow{RU} .
 (c) Calculate the size of angle TRU.



- (a) From the diagram, $T(0, 4, 4)$ and $U(4, 4, 4)$.

$$(b) \overrightarrow{RT} = \underline{t} - \underline{r} = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 4 \end{pmatrix},$$

$$\overrightarrow{RU} = \underline{u} - \underline{r} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}.$$

$$(c) \cos \widehat{TRU} = \frac{\overrightarrow{RT} \cdot \overrightarrow{RU}}{|\overrightarrow{RT}| |\overrightarrow{RU}|}$$

$$= \frac{(-4 \times 0) + (4 \times 4) + (4 \times 4)}{\sqrt{(-4)^2 + 4^2 + 4^2} \sqrt{0^2 + 4^2 + 4^2}}$$

$$= \frac{32}{\sqrt{3 \times 16} \sqrt{2 \times 16}}$$

$$= \frac{2}{\sqrt{6}}$$

$$\widehat{TRU} = \cos^{-1} \left(\frac{2}{\sqrt{6}} \right)$$

$$= 35.3^\circ \text{ (to 1 d.p.)} \quad (\text{or } 0.615 \text{ radians (to 3 d.p.)})$$

13 Perpendicular Vectors

If \underline{a} and \underline{b} are perpendicular then $\underline{a} \cdot \underline{b} = 0$.

$$\begin{aligned} \text{This is because } \underline{a} \cdot \underline{b} &= |\underline{a}| |\underline{b}| \cos \theta \\ &= |\underline{a}| |\underline{b}| \cos 90^\circ \quad (\theta = 90^\circ \text{ since perpendicular}) \\ &= 0 \quad (\text{since } \cos 90^\circ = 0). \end{aligned}$$

Conversely, if $\underline{a} \cdot \underline{b} = 0$ then \underline{a} and \underline{b} are perpendicular.

EXAMPLES

1. Two vectors are defined as $\underline{a} = 4\underline{i} + 2\underline{j} - 5\underline{k}$ and $\underline{b} = 2\underline{i} + \underline{j} + 2\underline{k}$.

Show that \underline{a} and \underline{b} are perpendicular.

$$\begin{aligned} \underline{a} \cdot \underline{b} &= a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= (4 \times 2) + (2 \times 1) + ((-5) \times 2) \\ &= 8 + 2 - 10 \\ &= 0. \end{aligned}$$

Since $\underline{a} \cdot \underline{b} = 0$, \underline{a} and \underline{b} are perpendicular.

2. $\overrightarrow{PQ} = \begin{pmatrix} 4 \\ a \\ 7 \end{pmatrix}$ and $\overrightarrow{RS} = \begin{pmatrix} 2 \\ -3 \\ a \end{pmatrix}$ where a is a constant.

Given that \overrightarrow{PQ} and \overrightarrow{RS} are perpendicular, find the value of a .

Since \overrightarrow{PQ} and \overrightarrow{RS} are perpendicular,

$$\begin{aligned} \overrightarrow{PQ} \cdot \overrightarrow{RS} &= 0 \\ 4 \times 2 + (-3a) + 7a &= 0 \\ 8 - 3a + 7a &= 0 \\ 8 + 4a &= 0 \\ a &= -2. \end{aligned}$$

14 Properties of the Scalar Product

Some useful properties of the scalar product are as follows:

$$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$$

$$\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c} \quad (\text{Expanding brackets})$$

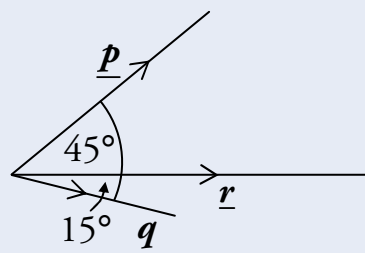
$$\underline{a} \cdot \underline{a} = |\underline{a}|^2.$$

Note that these are not given in the exam, so you need to remember them.

EXAMPLES

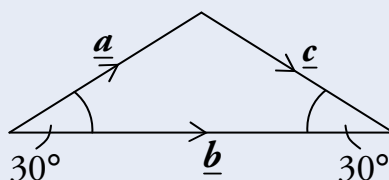
1. In the diagram, $|\underline{p}| = 3$, $|\underline{r}| = 4$ and $|\underline{q}| = 2$.

Calculate $\underline{p} \cdot (\underline{q} + \underline{r})$.



$$\begin{aligned} \underline{p} \cdot (\underline{q} + \underline{r}) &= \underline{p} \cdot \underline{q} + \underline{p} \cdot \underline{r} \\ &= |\underline{p}||\underline{q}|\cos\theta_1 + |\underline{p}||\underline{r}|\cos\theta_2 \\ &= 3 \times 2 \times \cos 60^\circ + 3 \times 4 \times \cos 45^\circ \\ &= 6 \times \frac{1}{2} + 12 \times \frac{1}{\sqrt{2}} \\ &= 3 + 6\sqrt{2}. \end{aligned}$$

2. In the diagram below, $|\underline{a}| = |\underline{c}| = 2$ and $|\underline{b}| = 2\sqrt{3}$.



Calculate $\underline{a} \cdot (\underline{a} + \underline{b} + \underline{c})$.

$$\begin{aligned} \underline{a} \cdot (\underline{a} + \underline{b} + \underline{c}) &= \underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c} \\ &= |\underline{a}|^2 + |\underline{a}||\underline{b}|\cos\theta_1 - |\underline{a}||\underline{c}|\cos\theta_2 \\ &= 2^2 + 2 \times 2\sqrt{3} \times \cos 30^\circ - 2 \times 2 \times \cos 120^\circ \\ &= 4 + 4\sqrt{3} \times \frac{\sqrt{3}}{2} + 4 \times \frac{1}{2} \\ &= 4 + 6 + 2 \\ &= 12. \end{aligned}$$

Remember

$\underline{a} \cdot \underline{c} = -|\underline{a}||\underline{c}|\cos\theta_2$
since \underline{a} points into θ_2
and \underline{c} points away.

OUTCOME 2

Further Calculus

1 Differentiating $\sin x$ and $\cos x$

In order to differentiate expressions involving trigonometric functions, we use the following rules:

$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x.$$

These rules only work when x is an angle measured in radians. A form of these rules is given in the exam.

EXAMPLES

1. Differentiate $y = 3 \sin x$ with respect to x .

$$\frac{dy}{dx} = 3 \cos x.$$

2. A function f is defined by $f(x) = \sin x - 2 \cos x$ for $x \in \mathbb{R}$.

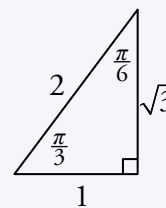
Find $f'\left(\frac{\pi}{3}\right)$.

$$\begin{aligned} f'(x) &= \cos x - (-2 \sin x) \\ &= \cos x + 2 \sin x \end{aligned}$$

$$\begin{aligned} f'\left(\frac{\pi}{3}\right) &= \cos \frac{\pi}{3} + 2 \sin \frac{\pi}{3} \\ &= \frac{1}{2} + 2 \times \frac{\sqrt{3}}{2} \\ &= \frac{1}{2} + \sqrt{3}. \end{aligned}$$

Remember

The exact value triangle:



3. Find the equation of the tangent to the curve $y = \sin x$ when $x = \frac{\pi}{6}$.

When $x = \frac{\pi}{6}$, $y = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. So the point is $\left(\frac{\pi}{6}, \frac{1}{2}\right)$.

We also need the gradient at the point where $x = \frac{\pi}{6}$:

$$\frac{dy}{dx} = \cos x.$$

$$\text{When } x = \frac{\pi}{6}, \quad m_{\text{tangent}} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.$$

Now we have the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ and the gradient $m_{\text{tangent}} = \frac{\sqrt{3}}{2}$, so:

$$y - b = m(x - a)$$

$$y - \frac{1}{2} = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$$

$$2y - 1 = \sqrt{3}x - \frac{\sqrt{3}\pi}{6}$$

$$\sqrt{3}x - 2y - \frac{\sqrt{3}\pi}{6} + 1 = 0.$$

2 Integrating $\sin x$ and $\cos x$

We know the derivatives of $\sin x$ and $\cos x$, so it follows that the integrals are:

$$\int \cos x \, dx = \sin x + c, \quad \int \sin x \, dx = -\cos x + c.$$

Again, these results only hold if x is measured in radians.

EXAMPLES

1. Find $\int (5 \sin x + 2 \cos x) \, dx$.

$$\int (5 \sin x + 2 \cos x) \, dx = -5 \cos x + 2 \sin x + c.$$

2. Find $\int_0^{\frac{\pi}{4}} (4 \cos x + 2 \sin x) \, dx$.

$$\begin{aligned} \int_0^{\frac{\pi}{4}} (4 \cos x + 2 \sin x) \, dx &= [4 \sin x - 2 \cos x]_0^{\frac{\pi}{4}} \\ &= \left(4 \sin\left(\frac{\pi}{4}\right) - 2 \cos\left(\frac{\pi}{4}\right)\right) - (4 \sin 0 - 2 \cos 0) \\ &= \left(\left(4 \times \frac{1}{\sqrt{2}}\right) - \left(2 \times \frac{1}{\sqrt{2}}\right)\right) - (-2) \\ &= \frac{4}{\sqrt{2}} - \frac{2}{\sqrt{2}} + 2 \\ &= \left(\frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\right) + 2 \\ &= \sqrt{2} + 2. \end{aligned}$$

Note

It is good practice to rationalise the denominator.



3. Find the value of $\int_0^4 \frac{1}{2} \sin x \, dx$.

$$\begin{aligned} \int_0^4 \frac{1}{2} \sin x \, dx &= \left[-\frac{1}{2} \cos x\right]_0^4 \\ &= -\frac{1}{2} \cos(4) + \frac{1}{2} \cos(0) \\ &= \frac{1}{2}(0.654 + 1) \\ &= 0.827 \text{ (to 3 d.p.)}. \end{aligned}$$

Remember

We must use radians when integrating or differentiating trig. functions.

3 The Chain Rule

We will now look at how to differentiate composite functions, such as $f(g(x))$. If the functions f and g are defined on suitable domains, then

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \times g'(x).$$

Stated simply: differentiate the outer function, the bracket stays the same, then multiply by the derivative of the bracket.

This is called the **chain rule**. You will need to remember it for the exam.

EXAMPLE

If $y = \cos\left(5x + \frac{\pi}{6}\right)$, find $\frac{dy}{dx}$.

$$y = \cos\left(5x + \frac{\pi}{6}\right)$$

$$\begin{aligned}\frac{dy}{dx} &= -\sin\left(5x + \frac{\pi}{6}\right) \times 5 \\ &= -5 \sin\left(5x + \frac{\pi}{6}\right).\end{aligned}$$

Note

The “ $\times 5$ ” comes from $\frac{d}{dx}\left(5x + \frac{\pi}{6}\right)$.

4 Special Cases of the Chain Rule

We will now look at how the chain rule can be applied to particular types of expression.

Powers of a Function

For expressions of the form $[f(x)]^n$, where n is a constant, we can use a simpler version of the chain rule:

$$\frac{d}{dx}[(f(x))^n] = n[f(x)]^{n-1} \times f'(x).$$

Stated simply: the power (n) multiplies to the front, the bracket stays the same, the power lowers by one (giving $n-1$) and everything is multiplied by the derivative of the bracket ($f'(x)$).

EXAMPLES

1. A function f is defined on a suitable domain by $f(x) = \sqrt{2x^2 + 3x}$.
Find $f'(x)$.

$$\begin{aligned} f(x) &= \sqrt{2x^2 + 3x} = (2x^2 + 3x)^{\frac{1}{2}} \\ f'(x) &= \frac{1}{2}(2x^2 + 3x)^{-\frac{1}{2}} \times (4x + 3) \\ &= \frac{1}{2}(4x + 3)(2x^2 + 3x)^{-\frac{1}{2}} \\ &= \frac{4x + 3}{2\sqrt{2x^2 + 3x}}. \end{aligned}$$

2. Differentiate $y = 2\sin^4 x$ with respect to x .

$$\begin{aligned} y &= 2\sin^4 x = 2(\sin x)^4 \\ \frac{dy}{dx} &= 2 \times 4(\sin x)^3 \times \cos x \\ &= 8\sin^3 x \cos x. \end{aligned}$$

Powers of a Linear Function

The rule for differentiating an expression of the form $(ax + b)^n$, where a , b and n are constants, is as follows:

$$\frac{d}{dx}[(ax + b)^n] = an(ax + b)^{n-1}.$$

EXAMPLES

3. Differentiate $y = (5x + 2)^3$ with respect to x .

$$\begin{aligned} y &= (5x + 2)^3 \\ \frac{dy}{dx} &= 3(5x + 2)^2 \times 5 \\ &= 15(5x + 2)^2. \end{aligned}$$

4. If $y = \frac{1}{(2x+6)^3}$, find $\frac{dy}{dx}$.

$$y = \frac{1}{(2x+6)^3} = (2x+6)^{-3}$$

$$\frac{dy}{dx} = -3(2x+6)^{-4} \times 2$$

$$= -6(2x+6)^{-4}$$

$$= -\frac{6}{(2x+6)^4}.$$

5. A function f is defined by $f(x) = \sqrt[3]{(3x-2)^4}$ for $x \in \mathbb{R}$. Find $f'(x)$.

$$f(x) = \sqrt[3]{(3x-2)^4} = (3x-2)^{\frac{4}{3}}$$

$$f'(x) = \frac{4}{3}(3x-2)^{\frac{1}{3}} \times 3$$

$$= \frac{4}{3} \sqrt[3]{3x-2}.$$

Trigonometric Functions

The following rules can be used to differentiate trigonometric functions.

$$\frac{d}{dx}[\sin(ax+b)] = a \cos(ax+b) \quad \frac{d}{dx}[\cos(ax+b)] = -a \sin(ax+b)$$

These are given in the exam.

EXAMPLE

6. Differentiate $y = \sin(9x + \pi)$ with respect to x .

$$\frac{dy}{dx} = 9 \cos(9x + \pi).$$

5 A Special Integral

The method for integrating an expression of the form $(ax + b)^n$ is:

$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + c \quad \text{where } a \neq 0 \text{ and } n \neq -1.$$

Stated simply: raise the power (n) by one, divide by the new power and also divide by the derivative of the bracket ($a(n+1)$), add c .

EXAMPLES

1. Find $\int (x + 4)^7 dx$.

$$\begin{aligned} \int (x + 4)^7 dx &= \frac{(x + 4)^8}{8 \times 1} + c \\ &= \frac{(x + 4)^8}{8} + c. \end{aligned}$$

2. Find $\int (2x + 3)^2 dx$.

$$\begin{aligned} \int (2x + 3)^2 dx &= \frac{(2x + 3)^3}{3 \times 2} + c \\ &= \frac{(2x + 3)^3}{6} + c. \end{aligned}$$

3. Find $\int \frac{1}{\sqrt[3]{5x+9}} dx$ where $x \neq -\frac{9}{5}$.

$$\begin{aligned} \int \frac{1}{\sqrt[3]{5x+9}} dx &= \int \frac{1}{(5x+9)^{\frac{1}{3}}} dx \\ &= \int (5x+9)^{-\frac{1}{3}} dx \\ &= \frac{(5x+9)^{\frac{2}{3}}}{\frac{2}{3} \times 5} + c \\ &= \frac{\sqrt[3]{5x+9}^2}{\frac{10}{3}} + c \\ &= \frac{3}{10} \sqrt[3]{5x+9}^2 + c. \end{aligned}$$



4. Evaluate $\int_0^3 \sqrt{3x+4} \, dx$ where $x \geq -\frac{4}{3}$.

$$\begin{aligned}
 \int_0^3 \sqrt{3x+4} \, dx &= \int_0^3 (3x+4)^{\frac{1}{2}} \, dx \\
 &= \left[\frac{(3x+4)^{\frac{3}{2}}}{\frac{3}{2} \times 3} \right]_0^3 \\
 &= \left[\frac{2}{9} \sqrt{(3x+4)^3} \right]_0^3 \\
 &= \frac{2}{9} \sqrt{(3 \times 3 + 4)^3} - \frac{2}{9} \sqrt{(3 \times 0 + 4)^3} \\
 &= \frac{2}{9} \sqrt{13^3} - \frac{2}{9} \sqrt{4^3} \\
 &= \frac{2}{9} (\sqrt{13^3} - 8) \quad (\text{or } 8.638 \text{ to 3 d.p.})
 \end{aligned}$$

Note

Changing powers back into roots here makes it easier to evaluate the two brackets.

Remember

To evaluate $\sqrt{4^3}$, it is easier to work out $\sqrt{4}$ first.

Warning

Make sure you don't confuse differentiation and integration – this could lose you a lot of marks in the exam.

Remember the following rules for differentiating and integrating expressions of the form $(ax+b)^n$:

$$\frac{d}{dx} [(ax+b)^n] = an(ax+b)^{n-1},$$

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c.$$

These rules will *not* be given in the exam.

Using Differentiation to Integrate

Recall that integration is just the process of undoing differentiation. So if we differentiate $f(x)$ to get $g(x)$ then we know that $\int g(x) dx = f(x) + c$.

EXAMPLES

5. (a) Differentiate $y = \frac{5}{(3x-1)^4}$ with respect to x .

(b) Hence, or otherwise, find $\int \frac{1}{(3x-1)^5} dx$.

$$(a) \quad y = \frac{5}{(3x-1)^4} = 5(3x-1)^{-4}$$

$$\begin{aligned} \frac{dy}{dx} &= 5 \times 3 \times (-4)(3x-1)^{-5} \\ &= -\frac{60}{(3x-1)^5}. \end{aligned}$$

(b) From part (a) we know $\int -\frac{60}{(3x-1)^5} dx = \frac{5}{(3x-1)^4} + c$. So:

$$-60 \int \frac{1}{(3x-1)^5} dx = \frac{5}{(3x-1)^4} + c$$

$$\int \frac{1}{(3x-1)^5} dx = -\frac{1}{60} \left(\frac{5}{(3x-1)^4} + c \right)$$

$$= -\frac{1}{12(3x-1)^4} + c_1 \quad \text{where } c_1 \text{ is some constant.}$$

Note

We could also have used the special integral to obtain this answer.

6. (a) Differentiate $y = \frac{1}{(x^3-1)^5}$ with respect to x .

(b) Hence, find $\int \frac{x^2}{(x^3-1)^6} dx$.

$$(a) \quad y = \frac{1}{(x^3-1)^5} = (x^3-1)^{-5}$$

$$\begin{aligned} \frac{dy}{dx} &= -5(x^3-1)^{-6} \times 3x^2 \\ &= -\frac{15x^2}{(x^3-1)^6}. \end{aligned}$$

(b) From part (a) we know $\int -\frac{15x^2}{(x^3-1)^6} dx = \frac{1}{(x^3-1)^5} + c$. So:

$$-15 \int \frac{x^2}{(x^3-1)^6} dx = \frac{1}{(x^3-1)^5} + c$$

$$\int \frac{x^2}{(x^3-1)^6} dx = -\frac{1}{15} \left(\frac{1}{(x^3-1)^5} + c \right)$$

$$= -\frac{1}{15(x^3-1)^5} + c_1 \quad \text{where } c_1 \text{ is some constant.}$$

Note

In this case, the special integral cannot be used.

6 Integrating $\sin(ax+b)$ and $\cos(ax+b)$

Since we know the derivatives of $\sin(ax+b)$ and $\cos(ax+b)$, it follows that their integrals are:

$$\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + c,$$

$$\int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b) + c.$$

These are given in the exam.

EXAMPLES

1. Find $\int \sin(4x+1) dx$.

$$\int \sin(4x+1) dx = -\frac{1}{4} \cos(4x+1) + c.$$

2. Find $\int \cos\left(\frac{3}{2}x + \frac{\pi}{5}\right) dx$.

$$\int \cos\left(\frac{3}{2}x + \frac{\pi}{5}\right) dx = \frac{2}{3} \sin\left(\frac{3}{2}x + \frac{\pi}{5}\right) + c.$$



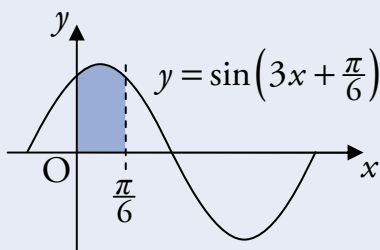
3. Find the value of $\int_0^1 \cos(2x-5) dx$.

$$\begin{aligned} \int_0^1 \cos(2x-5) dx &= \left[\frac{1}{2} \sin(2x-5) \right]_0^1 \\ &= \frac{1}{2} \sin(-3) - \frac{1}{2} \sin(-5) \\ &= \frac{1}{2} (-0.141 - 0.959) \\ &= -0.55 \quad (\text{to 2 d.p.}). \end{aligned}$$

Remember

We must use radians when integrating or differentiating trig. functions.

4. Find the area enclosed by the graph of $y = \sin\left(3x + \frac{\pi}{6}\right)$, the x -axis and the lines $x = 0$ and $x = \frac{\pi}{6}$.



$$\begin{aligned} \int_0^{\frac{\pi}{6}} \sin\left(3x + \frac{\pi}{6}\right) dx &= \left[-\frac{1}{3} \cos\left(3x + \frac{\pi}{6}\right)\right]_0^{\frac{\pi}{6}} \\ &= \left(-\frac{1}{3} \cos\left(3\left(\frac{\pi}{6}\right) + \frac{\pi}{6}\right)\right) - \left(-\frac{1}{3} \cos\left(3(0) + \frac{\pi}{6}\right)\right) \\ &= \left(\left(-\frac{1}{3}\right) \times \left(-\frac{1}{2}\right)\right) + \left(\frac{1}{3} \times \frac{\sqrt{3}}{2}\right) \\ &= \frac{1}{6} + \frac{\sqrt{3}}{6} \\ &= \frac{1 + \sqrt{3}}{6}. \end{aligned} \quad \text{So the area is } \frac{1 + \sqrt{3}}{6} \text{ square units.}$$

5. Find $\int 2 \cos\left(\frac{1}{2}x - 3\right) dx$.

$$\begin{aligned} \int 2 \cos\left(\frac{1}{2}x - 3\right) dx &= \frac{2}{\frac{1}{2}} \sin\left(\frac{1}{2}x - 3\right) + c \\ &= 4 \sin\left(\frac{1}{2}x - 3\right) + c. \end{aligned}$$

6. Find $\int (5 \cos(2x) + \sin(x - \sqrt{3})) dx$.

$$\int (5 \cos(2x) + \sin(x - \sqrt{3})) dx = \frac{5}{2} \sin(2x) - \cos(x - \sqrt{3}) + c.$$

7. (a) Differentiate $\frac{1}{\cos x}$ with respect to x .

(b) Hence find $\int \frac{\tan x}{\cos x} dx$.

$$\begin{aligned} \text{(a)} \quad \frac{1}{\cos x} &= (\cos x)^{-1}, \text{ and } \frac{d}{dx}(\cos x)^{-1} = -1(\cos x)^{-2} \times -\sin x \\ &= \frac{\sin x}{\cos^2 x}. \end{aligned}$$

$$\text{(b)} \quad \frac{\tan x}{\cos x} = \frac{\frac{\sin x}{\cos x}}{\cos x} = \frac{\sin x}{\cos^2 x}.$$

$$\text{From part (a) we know } \int \frac{\sin x}{\cos^2 x} dx = \frac{1}{\cos x} + c.$$

$$\text{Therefore } \int \frac{\tan x}{\cos x} dx = \frac{1}{\cos x} + c.$$

OUTCOME 3

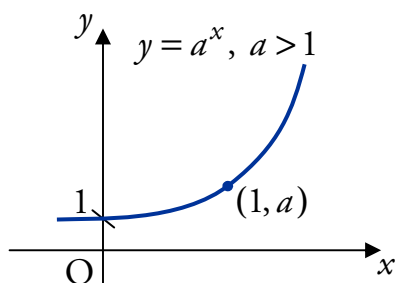
Exponentials and Logarithms

1 Exponentials

We have already met exponential functions in Unit 1 Outcome 2.

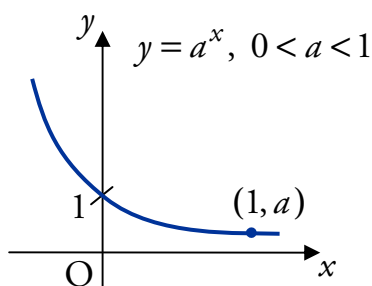
A function of the form $f(x) = a^x$ where $a, x \in \mathbb{R}$ and $a > 0$ is known as an **exponential function** to the base a .

If $a > 1$ then the graph looks like this:



This is sometimes called a **growth** function.

If $0 < a < 1$ then the graph looks like this:



This is sometimes called a **decay** function.

Remember that the graph of an exponential function $f(x) = a^x$ always passes through (0, 1) and (1, a) since

$$f(0) = a^0 = 1, \quad f(1) = a^1 = a.$$

EXAMPLES



1. The otter population on an island increases by 16% per year. How many full years will it take for the population to double?

Let u_0 be the initial population.

$$u_1 = 1.16u_0 \quad (116\% \text{ as a decimal})$$

$$u_2 = 1.16u_1 = 1.16(1.16u_0) = 1.16^2 u_0$$

$$u_3 = 1.16u_2 = 1.16(1.16^2 u_0) = 1.16^3 u_0$$

$$\vdots$$

$$u_n = 1.16^n u_0.$$

For the population to double after n years, we require $u_n \geq 2u_0$.

We want to know the smallest n which gives 1.16^n a value of 2 or more, since this will make u_n at least twice as big as u_0 .

Try values of n until this is satisfied.

If $n = 2$, $1.16^2 = 1.35 < 2$ On a calculator:

1	.	1	6	=
---	---	---	---	---

If $n = 3$, $1.16^3 = 1.56 < 2$

1	.	1	6	ANS	=
---	---	---	---	-----	---

If $n = 4$, $1.16^4 = 1.81 < 2$

=

If $n = 5$, $1.16^5 = 2.10 > 2$

⋮

Therefore after 5 years the population will double.



2. The efficiency of a machine decreases by 5% each year. When the efficiency drops below 75%, the machine needs to be serviced.

After how many years will the machine need serviced?

Let u_0 be the initial efficiency.

$$u_1 = 0.95u_0 \quad (95\% \text{ as a decimal})$$

$$u_2 = 0.95u_1 = 0.95(0.95u_0) = 0.95^2 u_0$$

$$u_3 = 0.95u_2 = 0.95(0.95^2 u_0) = 0.95^3 u_0$$

$$\vdots$$

$$u_n = 0.95^n u_0$$

When the efficiency drops below $0.75u_0$ (75% of the initial value) the machine must be serviced. So the machine needs serviced after n years if $0.95^n \leq 0.75$.

Try values of n until this is satisfied.

$$\text{If } n = 2, 0.95^2 = 0.903 > 0.75$$

$$\text{If } n = 3, 0.95^3 = 0.857 > 0.75$$

$$\text{If } n = 4, 0.95^4 = 0.815 > 0.75$$

$$\text{If } n = 5, 0.95^5 = 0.774 > 0.75$$

$$\text{If } n = 6, 0.95^6 = 0.735 < 0.75$$

Therefore after 6 years, the machine will have to be serviced.

2 Logarithms

Having previously defined what a logarithm is (see Unit 1 Outcome 2) we want to look in more detail at the properties of these important functions.

The relationship between logarithms and exponentials is expressed as:

$$y = \log_a x \Leftrightarrow x = a^y \quad \text{where } a, x > 0.$$

Here, y is the power of a which gives x .

EXAMPLES

1. Write $5^3 = 125$ in logarithmic form.

$$5^3 = 125 \Leftrightarrow 3 = \log_5 125.$$

2. Evaluate $\log_4 16$.

The power of 4 which gives 16 is 2, so $\log_4 16 = 2$.

3 Laws of Logarithms

There are three laws of logarithms which you must know.

Rule 1

$$\log_a x + \log_a y = \log_a (xy) \quad \text{where } a, x, y > 0.$$

If two logarithmic terms with the same base number (a above) are being added together, then the terms can be combined by multiplying the arguments (x and y above).

EXAMPLE

1. Simplify $\log_5 2 + \log_5 4$.

$$\begin{aligned} & \log_5 2 + \log_5 4 \\ &= \log_5 (2 \times 4) \\ &= \log_5 8. \end{aligned}$$

Rule 2

$$\log_a x - \log_a y = \log_a \left(\frac{x}{y} \right) \quad \text{where } a, x, y > 0.$$

If a logarithmic term is being subtracted from another logarithmic term with the same base number (a above), then the terms can be combined by dividing the arguments (x and y in this case).

Note that the argument which is being taken away (y above) appears on the bottom of the fraction when the two terms are combined.

EXAMPLE

2. Evaluate $\log_4 6 - \log_4 3$.

$$\begin{aligned} & \log_4 6 - \log_4 3 \\ &= \log_4 \left(\frac{6}{3} \right) \\ &= \log_4 2 \\ &= \frac{1}{2} \quad (\text{since } 4^{\frac{1}{2}} = \sqrt{4} = 2). \end{aligned}$$

Rule 3

$$\log_a x^n = n \log_a x \quad \text{where } a, x > 0.$$

The power of the argument (n above) can come to the front of the term as a multiplier, and vice-versa.

EXAMPLE

3. Express $2 \log_7 3$ in the form $\log_7 a$.

$$\begin{aligned} & 2 \log_7 3 \\ &= \log_7 3^2 \\ &= \log_7 9. \end{aligned}$$

Squash, Split and Fly

You may find the following names are a simpler way to remember the laws of logarithms.

- $\log_a x + \log_a y = \log_a (xy)$ – the arguments are **squashed** together by multiplying.
- $\log_a x - \log_a y = \log_a \left(\frac{x}{y} \right)$ – the arguments are **split** into a fraction.
- $\log_a x^n = n \log_a x$ – the power of an argument can **fly** to the front of the log term and vice-versa.

Note

When working with logarithms, you should remember:

$$\log_a 1 = 0 \quad \text{since } a^0 = 1,$$

$$\log_a a = 1 \quad \text{since } a^1 = a.$$

EXAMPLE

4. Evaluate $\log_7 7 + \log_3 3$.

$$\begin{aligned} \log_7 7 + \log_3 3 \\ &= 1 + 1 \\ &= 2. \end{aligned}$$

Combining several log terms

When adding and subtracting several log terms in the form $\log_a b$, there is a simple way to combine all the terms in one step.

$$\log_a \left(\frac{\quad}{\quad} \right)$$

- Multiply the arguments of the positive log terms in the numerator.
- Multiply the arguments of the negative log terms in the denominator.

EXAMPLES

5. Evaluate $\log_{12} 10 + \log_{12} 6 - \log_{12} 5$

$$\begin{aligned} \log_{12} 10 + \log_{12} 6 - \log_{12} 5 \\ &= \log_{12} \left(\frac{10 \times 6}{5} \right) \\ &= \log_{12} 12 \\ &= 1. \end{aligned}$$

$$\log_{12} \left(\frac{\quad}{\quad} \right)$$

6. Evaluate $\log_6 4 + 2 \log_6 3$.

$$\begin{aligned} \log_6 4 + 2 \log_6 3 \\ &= \log_6 4 + \log_6 3^2 \\ &= \log_6 4 + \log_6 9 \\ &= \log_6 (4 \times 9) \\ &= \log_6 36 \\ &= 2 \quad (\text{since } 6^2 = 36). \end{aligned}$$

OR

$$\begin{aligned} \log_6 4 + 2 \log_6 3 \\ &= \log_6 2^2 + 2 \log_6 3 \\ &= 2 \log_6 2 + 2 \log_6 3 \\ &= 2(\log_6 2 + \log_6 3) \\ &= 2(\log_6 (2 \times 3)) \\ &= 2 \log_6 6 \\ &= 2 \quad (\text{since } \log_6 6 = 1). \end{aligned}$$

4 Exponentials and Logarithms to the Base e

The constant e is an important number in Mathematics, and occurs frequently in models of real-life situations. Its value is roughly 2.718281828 (to 9 d.p.), and is defined as:

$$\left(1 + \frac{1}{n}\right)^n \text{ as } n \rightarrow \infty.$$

If you try large values of n on your calculator, you will get close to the value of e . Like π , e is an irrational number.

Throughout this section, we will use e in expressions of the form:

- e^x , which is called an exponential to the base e ,
- $\log_e x$, which is called a logarithm to the base e . This is also known as the **natural logarithm** of x , and is often written as $\ln x$ (i.e. $\ln x \equiv \log_e x$).

EXAMPLES



1. Calculate the value of $\log_e 8$.

$$\log_e 8 = 2.08 \text{ (to 2 d.p.)}. \quad \text{On a calculator: } \boxed{\ln} \boxed{8} \boxed{=}$$



2. Solve $\log_e x = 9$.

$$\begin{aligned} \log_e x &= 9 \\ \text{so } x &= e^9 \quad \text{On a calculator: } \boxed{e^x} \boxed{9} \boxed{=} \\ x &= 8103.08 \text{ (to 2 d.p.)}. \end{aligned}$$

3. Simplify $4\log_e(2e) - 3\log_e(3e)$ expressing your answer in the form $a + \log_e b - \log_e c$ where a , b and c are whole numbers.

$$\begin{aligned} &4\log_e(2e) - 3\log_e(3e) \\ &= 4\log_e 2 + \underbrace{4\log_e e}_1 - 3\log_e 3 - \underbrace{3\log_e e}_1 \\ &= 4\log_e 2 + 4 - 3\log_e 3 - 3 \\ &= 1 + 4\log_e 2 - 3\log_e 3 \\ &= 1 + \log_e 2^4 - \log_e 3^3 \\ &= 1 + \log_e 16 - \log_e 27. \end{aligned}$$

$$\begin{aligned} \text{OR } &4\log_e(2e) - 3\log_e(3e) \\ &= \log_e(2e)^4 - \log_e(3e)^3 \\ &= \log_e \left(\frac{(2e)^4}{(3e)^3} \right) \\ &= \log_e \left(\frac{16e^4}{27e^3} \right) \\ &= \log_e \left(\frac{16e}{27} \right) \\ &= \log_e e + \log_e 16 - \log_e 27 \\ &= 1 + \log_e 16 - \log_e 27. \end{aligned}$$

Remember

$$(ab)^n = a^n b^n.$$

5 Exponential and Logarithmic Equations

Many mathematical models of real-life situations use exponentials and logarithms. It is important to become familiar with using the laws of logarithms to help solve equations.

EXAMPLES

1. Solve $\log_a 13 + \log_a x = \log_a 273$ for $x > 0$.

$$\log_a 13 + \log_a x = \log_a 273$$

$$\log_a 13x = \log_a 273$$

$$13x = 273 \quad (\text{since } \log_a x = \log_a y \Leftrightarrow x = y)$$

$$x = 21.$$

2. Solve $\log_{11}(4x+3) - \log_{11}(2x-3) = 1$ for $x > \frac{3}{2}$.

$$\log_{11}(4x+3) - \log_{11}(2x-3) = 1$$

$$\log_{11}\left(\frac{4x+3}{2x-3}\right) = 1$$

$$\frac{4x+3}{2x-3} = 11^1 = 11 \quad (\text{since } \log_a x = y \Leftrightarrow x = a^y)$$

$$4x+3 = 11(2x-3)$$

$$4x+3 = 22x-33$$

$$18x = 36$$

$$x = 2.$$

3. Solve $\log_a(2p+1) + \log_a(3p-10) = \log_a(11p)$ for $p > 4$.

$$\log_a(2p+1) + \log_a(3p-10) = \log_a(11p)$$

$$\log_a((2p+1)(3p-10)) = \log_a(11p)$$

$$(2p+1)(3p-10) = 11p$$

$$6p^2 - 20p + 3p - 10 - 11p = 0$$

$$6p^2 - 28p - 10 = 0$$

$$(3p+1)(p-5) = 0$$

$$3p+1=0 \quad \text{or} \quad p-5=0$$

$$p = -\frac{1}{3} \quad p = 5.$$

Since we require $p > 4$, $p = 5$ is the solution.

Dealing with Constants

Sometimes it may be necessary to write constants as logs, in order to solve equations.

EXAMPLE

4. Solve $\log_2 7 = \log_2 x + 3$ for $x > 0$.

Write 3 in logarithmic form:

$$\begin{aligned} 3 &= 3 \times 1 \\ &= 3 \log_2 2 \quad (\text{since } \log_2 2 = 1) \\ &= \log_2 2^3 \\ &= \log_2 8. \end{aligned}$$

Use this in the equation:

$$\begin{aligned} \log_2 7 &= \log_2 x + \log_2 8 \\ \log_2 7 &= \log_2 8x \\ 7 &= 8x \\ x &= \frac{7}{8}. \end{aligned}$$

OR $\log_2 7 = \log_2 x + 3$

$$\begin{aligned} \log_2 7 - \log_2 x &= 3 \\ \log_2 \left(\frac{7}{x} \right) &= 3. \end{aligned}$$

Converting from log to exponential form:

$$\begin{aligned} \frac{7}{x} &= 2^3 \\ x &= \frac{7}{2^3} = \frac{7}{8}. \end{aligned}$$

Solving Equations with Unknown Exponents

If an unknown value (e.g. x) is the power of a term (e.g. e^x or 10^x), and its value is to be calculated, then we must take logs on both sides of the equation to allow it to be solved.

The same solution will be reached using any base, but calculators can be used for evaluating logs either in base e or base 10.

EXAMPLES



5. Solve $e^x = 7$.

Taking \log_e of both sides

$$\begin{aligned} \log_e e^x &= \log_e 7 \\ x \log_e e &= \log_e 7 \quad (\log_e e = 1) \\ x &= \log_e 7 \\ x &= 1.946 \quad (\text{to 3 d.p.}). \end{aligned}$$

OR Taking \log_{10} of both sides

$$\begin{aligned} \log_{10} e^x &= \log_{10} 7 \\ x \log_{10} e &= \log_{10} 7 \\ x &= \frac{\log_{10} 7}{\log_{10} e} \\ x &= 1.946 \quad (\text{to 3 d.p.}). \end{aligned}$$



6. Solve $5^{3x+1} = 40$.

$$\log_e 5^{3x+1} = \log_e 40$$

$$(3x+1)\log_e 5 = \log_e 40$$

$$3x+1 = \frac{\log_e 40}{\log_e 5}$$

$$3x+1 = 2.2920$$

$$3x = 1.2920$$

$$x = 0.431 \text{ (to 3 d.p.)}$$

Note

\log_{10} could have been used instead of \log_e .

Exponential Growth and Decay

Recall from Section 1 that exponential functions are sometimes known as growth or decay functions. These often occur in models of real-life situations.

For instance, radioactive decay can be modelled using an exponential function. An important measurement is the **half-life** of a radioactive substance, which is the time taken for the mass of the radioactive substance to halve.

7. The mass G grams of a radioactive sample after time t years is given by the formula $G = 100e^{-3t}$.

(a) What was the initial mass of radioactive substance in the sample?

(b) Find the half-life of the radioactive substance.



(a) The initial mass was present when $t = 0$:

$$G = 100e^{-3 \times 0}$$

$$= 100e^0$$

$$= 100.$$

So the initial mass was 100 grams.

(b) The half-life is the time t at which $G = 50$, so

$$50 = 100e^{-3t}$$

$$e^{-3t} = \frac{50}{100} = \frac{1}{2}$$

$$-3t = \log_e \left(\frac{1}{2} \right) \quad (\text{converting to log form})$$

$$t = 0.231 \text{ (to 3 d.p.)}$$

So the half-life is 0.231 years, roughly $0.231 \times 356 = 84.315$ days.

8. The world population, in billions, t years after 1950 is given by

$$P = 2.54e^{0.0178t}.$$

- (a) What was the world population in 1950?
 (b) Find, to the nearest year, the time taken for the world population to double.



(a) For 1950, $t = 0$:

$$\begin{aligned} P &= 2.54e^{0.0178 \times 0} \\ &= 2.54e^0 \\ &= 2.54. \end{aligned}$$

So the world population in 1950 was 2.54 billion.

(b) For the population to double:

$$\begin{aligned} 2.54e^{0.0178t} &= 2 \times 2.54 \\ e^{0.0178t} &= 2 \\ 0.0178t &= \log_e 2 \quad (\text{converting to log form}) \\ t &= 38.94 \quad (\text{to 2 d.p.}). \end{aligned}$$

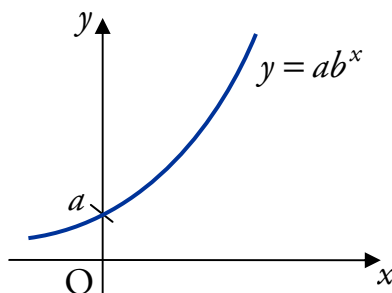
So the population doubled after 39 years (to the nearest year).

6 Graphing with Logarithmic Axes

It is common in applications to find an exponential relationship between variables; for instance, the relationship between the world population and time in the previous example. Given some data (e.g. from an experiment) we would like to find an explicit equation for the relationship.

Relationships of the form $y = ab^x$

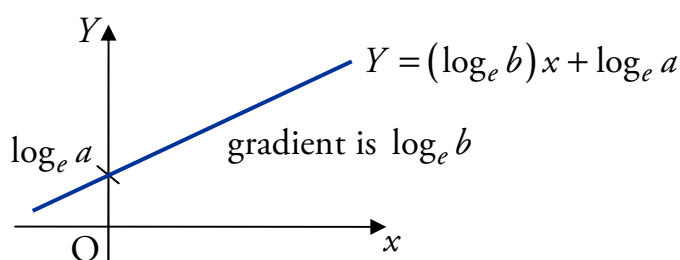
Suppose we have an exponential graph $y = ab^x$, where $a, b > 0$.



Taking logarithms we find that

$$\begin{aligned}\log_e y &= \log_e (ab^x) \\ &= \log_e a + \log_e b^x \\ &= \log_e a + x \log_e b.\end{aligned}$$

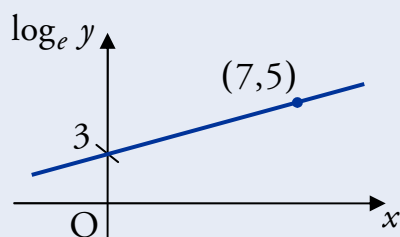
We can scale the y -axis so that $Y = \log_e y$; the Y -axis is called a **logarithmic axis**. Now our relationship is of the form $Y = (\log_e b)x + \log_e a$, which is a straight line in the (x, Y) -plane.



Since this is just a straight line, we can use known points to find the gradient $\log_e b$ and the Y -axis intercept $\log_e a$. From these we can easily find the values of a and b , and hence specify the equation $y = ab^x$.

EXAMPLES

- The relationship between two variables, x and y , is of the form $y = ab^x$. An experiment to test this relationship produced the data shown in the graph, where $\log_e y$ is plotted against x .



Find the values of a and b .

We need to obtain a straight line equation:

$$y = ab^x$$

$$\log_e y = \log_e ab^x \quad (\text{taking logs of both sides})$$

$$\log_e y = \log_e a + \log_e b^x$$

$$\log_e y = \log_e a + x \log_e b$$

$$\text{i.e. } Y = (\log_e b)x + \log_e a.$$

From the graph, the Y -axis intercept is $\log_e a = 3$; so $a = e^3$.

Using the gradient formula:

$$\begin{aligned}\log_e b &= \frac{5-3}{7-0} \\ &= \frac{2}{7} \\ b &= e^{\frac{2}{7}}.\end{aligned}$$

2. The results from an experiment were noted as follows:

x	1.30	2.00	2.30	2.80
$\log_e y$	2.04	2.56	2.78	3.14

The relationship between these data can be written in the form $y = ab^x$.



Find the values of a and b , and state the formula for y in terms of x .

We need to obtain a straight line equation:

$$y = ab^x$$

$$\log_e y = \log_e ab^x \quad (\text{taking logs of both sides})$$

$$\log_e y = \log_e a + \log_e b^x$$

$$\log_e y = \log_e a + x \log_e b$$

$$\text{i.e. } Y = (\log_e b)x + \log_e a.$$

We can find the gradient $\log_e b$ (and hence b), using two points on the line:

$$\begin{aligned}\text{using } (1.30, 2.04) \text{ and } (2.80, 3.14), \log_e b &= \frac{3.14 - 2.04}{2.80 - 1.30} \\ &= 0.73 \quad (\text{to 2 d.p.})\end{aligned}$$

$$\text{So } b = e^{0.73} = 2.08 \quad (\text{to 2 d.p.}).$$

$$\text{So } \log_e y = 0.73x + \log_e a.$$

Now we can work out $\log_e a$ (and hence a) by substituting a point into this equation:

$$\text{using } (1.30, 2.04), \log_e y = 2.04 \text{ and } x = 1.30$$

$$\text{so } 2.04 = 0.73 \times 1.30 + \log_e a$$

$$\log_e a = 2.04 - 0.73 \times 1.30$$

$$= 1.09 \quad (\text{to 2 d.p.})$$

$$\text{so } a = e^{1.09}$$

$$= 2.97 \quad (\text{to 2 d.p.}).$$

Therefore $y = 2.97 \times 2.08^x$.

Note

Depending on the points used, slightly different values for a and b may be obtained.

Equations in the form $y = ax^b$

Another common relationship is $y = ax^b$, where $a, x > 0$. In this case, the relationship can be represented by a straight line if we change *both* axes to logarithmic ones.

EXAMPLE

3. The results from an experiment were noted as follows:

$\log_{10} x$	1.70	2.29	2.70	2.85
$\log_{10} y$	1.33	1.67	1.92	2.01

The relationship between these data can be written in the form $y = ax^b$.

Find the values of a and b , and state the formula for y in terms of x .



We need to obtain a straight line equation:

$$y = ax^b$$

$$\log_{10} y = \log_{10} ax^b \quad (\text{taking logs of both sides})$$

$$\log_{10} y = \log_{10} a + \log_{10} x^b$$

$$\log_{10} y = \log_{10} a + b \log_{10} x$$

$$\text{i.e. } Y = bX + \log_{10} a.$$

We can find the gradient b using two points on the line:

$$\begin{aligned} \text{using } (1.70, 1.33) \text{ and } (2.85, 2.01), \quad b &= \frac{2.01 - 1.33}{2.85 - 1.70} \\ &= 0.59 \quad (\text{to 2 d.p.}). \end{aligned}$$

$$\text{So } \log_{10} y = 0.59 \log_{10} x + \log_{10} a.$$

Now we can work out a by substituting a point into this equation:

$$\text{using } (1.70, 1.33), \quad 1.33 = 0.59 \times 1.70 + \log_{10} a$$

$$\log_{10} a = 1.33 - 0.59 \times 1.70$$

$$= 0.33$$

$$a = 10^{0.33}$$

$$= 2.14 \quad (\text{to 2 d.p.}).$$

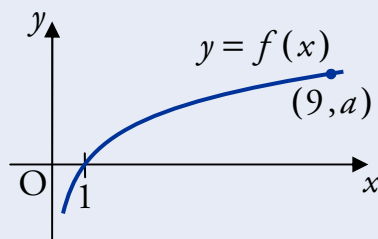
Therefore $y = 2.14x^{0.59}$.

7 Graph Transformations

Graph transformations were covered in Unit 1 Outcome 2 – Functions and Graphs, but we will now look in more detail at applying transformations to graphs of exponential and logarithmic functions.

EXAMPLES

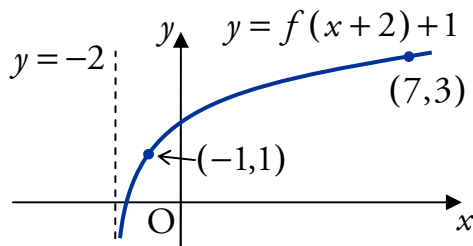
1. Shown below is the graph of $y = f(x)$ where $f(x) = \log_3 x$.



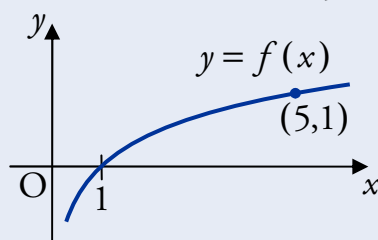
- (a) State the value of a .
 (b) Sketch the graph of $y = f(x+2)+1$.

(a) $a = \log_3 9$
 $= 2$ (since $3^2 = 9$).

- (b) The graph shifts two units to the left, and one unit upwards:



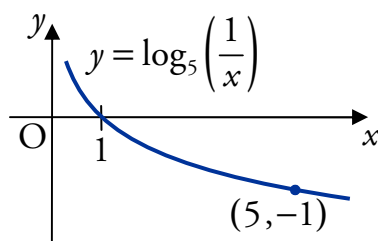
2. Shown below is part of the graph of $y = \log_5 x$.



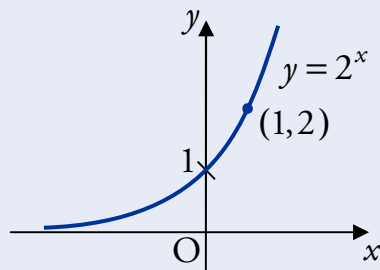
Sketch the graph of $y = \log_5 \left(\frac{1}{x} \right)$.

$$\begin{aligned} y &= \log_5 \left(\frac{1}{x} \right) \\ &= \log_5 x^{-1} \\ &= -\log_5 x. \end{aligned}$$

So reflect in the x -axis.



3. The diagram shows the graph of $y = 2^x$.

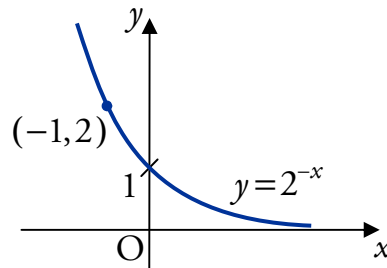


On separate diagrams, sketch the graphs of:

(a) $y = 2^{-x}$;

(b) $y = 2^{2-x}$.

(a) Reflect in the y -axis:

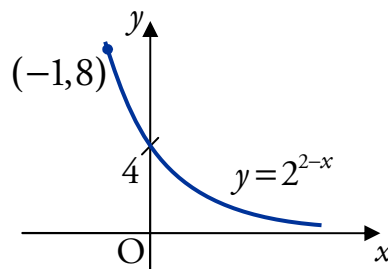


(b) $y = 2^{2-x}$

$$= 2^2 2^{-x}$$

$$= 4 \times 2^{-x}.$$

So scale the graph from (a) by 4 in the y -direction:



OUTCOME 4

Wave Functions

1 Expressing $p\cos x + q\sin x$ in the form $k\cos(x - a)$

An expression of the form $p\cos x + q\sin x$ can be written in the form $k\cos(x - a)$ where

$$k = \sqrt{p^2 + q^2} \quad \text{and} \quad \tan a = \frac{k \sin a}{k \cos a}.$$

The following example shows how to achieve this.

EXAMPLES



1. Write $5\cos x^\circ + 12\sin x^\circ$ in the form $k\cos(x^\circ - a^\circ)$ where $0 \leq a \leq 360$.

Step 1

Expand $k\cos(x - a)$ using the compound angle formula.

$$\begin{aligned} 5\cos x^\circ + 12\sin x^\circ \\ &= k\cos(x^\circ - a^\circ) \\ &= k\cos x^\circ \cos a^\circ + k\sin x^\circ \sin a^\circ \end{aligned}$$

Step 2

Rearrange to compare with $p\cos x + q\sin x$.

$$= \underbrace{(k\cos a^\circ)}_5 \cos x^\circ + \underbrace{(k\sin a^\circ)}_{12} \sin x^\circ$$

Step 3

Compare the coefficients of $\cos x$ and $\sin x$ with $p\cos x + q\sin x$.

$$\begin{aligned} k\cos a^\circ &= 5 \\ k\sin a^\circ &= 12 \end{aligned}$$

Step 4

Mark the quadrants on a CAST diagram, according to the signs of $k\cos a$ and $k\sin a$.

$$\begin{array}{c|c} 180^\circ - a^\circ & \begin{array}{l} \checkmark S \\ A \checkmark \checkmark \end{array} \\ \hline 180^\circ + a^\circ & \begin{array}{l} T \checkmark \\ C \checkmark \end{array} \\ & 360^\circ - a^\circ \end{array}$$

Step 5

Find k and a using the formulae above (a lies in the quadrant marked twice in Step 4).

$$\begin{aligned} k &= \sqrt{5^2 + 12^2} & \tan a^\circ &= \frac{k \sin a^\circ}{k \cos a^\circ} \\ &= \sqrt{169} & &= \frac{12}{5} \\ &= 13 & a &= \tan^{-1}\left(\frac{12}{5}\right) \\ & & &= 67.4 \quad (\text{to 1 d.p.}) \end{aligned}$$

Step 6

State $p\cos x + q\sin x$ in the form $k\cos(x - a)$ using these values.

$$5\cos x^\circ + 12\sin x^\circ = 13\cos(x^\circ - 67.4^\circ)$$



2. Write $5 \cos x - 3 \sin x$ in the form $k \cos(x - a)$ where $0 \leq a \leq 2\pi$.

$$\begin{aligned} 5 \cos x - 3 \sin x &= k \cos(x - a) \\ &= k \cos x \cos a + k \sin x \sin a \\ &= (k \cos a) \cos x + (k \sin a) \sin x. \end{aligned}$$

$$k \cos a = 5 \qquad k = \sqrt{5^2 + (-3)^2} \qquad \tan a = \frac{k \sin a}{k \cos a} = -\frac{3}{5}$$

$$k \sin a = -3 \qquad = \sqrt{34}$$

First quadrant answer is:

$$\begin{array}{c|c} \pi - a & \text{S} \mid \text{A} \checkmark \\ \hline \checkmark \text{T} & \text{C} \checkmark \checkmark \\ \pi + a & \text{T} \mid \text{C} \checkmark \checkmark \\ & 2\pi - a \end{array}$$

$$\tan^{-1}\left(\frac{3}{5}\right)$$

$$= 0.540 \quad (\text{to 3 d.p.})$$

$$\text{So } a = 2\pi - 0.540$$

$$= 5.743 \quad (\text{to 3 d.p.})$$

Note

Make sure your calculator is in radian mode.

Hence a is in the fourth quadrant.

$$\text{Hence } 5 \cos x - 3 \sin x = \sqrt{34} \cos(x - 5.743).$$

2 Expressing $p \cos x + q \sin x$ in other forms

An expression in the form $p \cos x + q \sin x$ can also be written in any of the following forms using a similar method:

$$k \cos(x + a), \quad k \sin(x - a), \quad k \sin(x + a).$$

EXAMPLES



1. Write $4 \cos x^\circ + 3 \sin x^\circ$ in the form $k \sin(x^\circ + a^\circ)$ where $0 \leq a \leq 360$.

$$\begin{aligned} 4 \cos x^\circ + 3 \sin x^\circ &= k \sin(x^\circ + a^\circ) \\ &= k \sin x^\circ \cos a^\circ + k \cos x^\circ \sin a^\circ \\ &= (k \cos a^\circ) \sin x^\circ + (k \sin a^\circ) \cos x^\circ \end{aligned}$$

$$k \cos a^\circ = 3 \qquad k = \sqrt{4^2 + 3^2} \qquad \tan a^\circ = \frac{k \sin a^\circ}{k \cos a^\circ} = \frac{4}{3}$$

$$k \sin a^\circ = 4 \qquad = \sqrt{25}$$

So:

$$\begin{array}{c|c} 180^\circ - a^\circ & \text{S} \mid \text{A} \checkmark \checkmark \\ \hline \checkmark \text{T} & \text{C} \checkmark \\ 180^\circ + a^\circ & \text{T} \mid \text{C} \checkmark \\ & 360^\circ - a^\circ \end{array}$$

$$a = \tan^{-1}\left(\frac{4}{3}\right)$$

$$= 53.1 \quad (\text{to 1 d.p.})$$

Hence a is in the first quadrant.

$$\text{Hence } 4 \cos x^\circ + 3 \sin x^\circ = 5 \sin(x^\circ + 53.1^\circ).$$

2. Write $\cos x - \sqrt{3} \sin x$ in the form $k \cos(x + a)$ where $0 \leq a \leq 2\pi$.

$$\begin{aligned}\cos x - \sqrt{3} \sin x &= k \cos(x + a) \\ &= k \cos x \cos a - k \sin x \sin a \\ &= (k \cos a) \cos x - (k \sin a) \sin x\end{aligned}$$

$$\begin{array}{lll}k \cos a = 1 & k = \sqrt{1^2 + (-\sqrt{3})^2} & \tan a = \frac{k \sin a}{k \cos a} = \sqrt{3} \\k \sin a = \sqrt{3} & = \sqrt{1+3} & \text{So:} \\ \pi - a & = \sqrt{4} & a = \tan^{-1}(\sqrt{3}) \\ \checkmark S | A \checkmark \checkmark & = 2 & = \frac{\pi}{3} \\ \pi + a & \checkmark T | C \checkmark & \\ \checkmark & & \checkmark \\ 2\pi - a & & \checkmark\end{array}$$

Hence a is in the first quadrant.

$$\text{Hence } \cos x - \sqrt{3} \sin x = 2 \cos\left(x + \frac{\pi}{3}\right).$$

3 Multiple Angles

We can use the same method with expressions involving the same multiple angle, i.e. $p \cos(nx) + q \sin(nx)$, where n is a constant.

EXAMPLE



Write $5 \cos 2x^\circ + 12 \sin 2x^\circ$ in the form $k \sin(2x^\circ + a^\circ)$ where $0 \leq a \leq 360$.

$$\begin{aligned}5 \cos 2x^\circ + 12 \sin 2x^\circ &= k \sin(2x^\circ + a^\circ) \\ &= k \sin 2x^\circ \cos a^\circ + k \cos 2x^\circ \sin a^\circ \\ &= (k \cos a^\circ) \sin 2x^\circ + (k \sin a^\circ) \cos 2x^\circ\end{aligned}$$

$$\begin{array}{lll}k \cos a^\circ = 12 & k = \sqrt{12^2 + 5^2} & \tan a^\circ = \frac{k \sin a^\circ}{k \cos a^\circ} = \frac{5}{12} \\k \sin a^\circ = 5 & = \sqrt{169} & \text{So:} \\ 180^\circ - a^\circ & = 13 & a = \tan^{-1}\left(\frac{5}{12}\right) \\ \checkmark S | A \checkmark \checkmark & & = 22.6 \text{ (to 1 d.p.)} \\ 180^\circ + a^\circ & \checkmark T | C \checkmark & \\ \checkmark & & \checkmark \\ 360^\circ - a^\circ & & \checkmark\end{array}$$

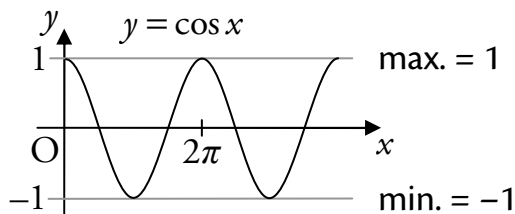
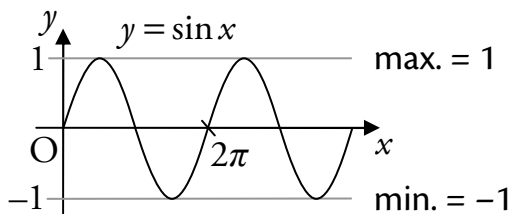
Hence a is in the first quadrant.

$$\text{Hence } 5 \cos 2x^\circ + 12 \sin 2x^\circ = 13 \sin(2x^\circ + 22.6^\circ).$$

4 Maximum and Minimum Values

To work out the maximum or minimum values of $p \cos x + q \sin x$, we can rewrite it as a single trigonometric function, e.g. $k \cos(x - a)$.

Recall that the maximum value of the sine and cosine functions is 1, and their minimum is -1 .



EXAMPLE

Write $4 \sin x + \cos x$ in the form $k \cos(x - a)$ where $0 \leq a < 2\pi$ and state:

- (i) the maximum value and the value of $0 \leq x < 2\pi$ at which it occurs
 (ii) the minimum value and the value of $0 \leq x < 2\pi$ at which it occurs.



$$\begin{aligned} 4 \sin x + \cos x &= k \cos(x - a) \\ &= k \cos x \cos a + k \sin x \sin a \\ &= (k \cos a) \cos x + (k \sin a) \sin x \end{aligned}$$

$$k \cos a = 1$$

$$k \sin a = 4$$

$$\begin{array}{c|c} \pi - a & A \\ \hline \sqrt{S} & \sqrt{A} \\ \hline \pi + a & T \\ \hline & C \\ & 2\pi - a \end{array}$$

Hence a is in the first quadrant.

$$\text{Hence } 4 \sin x + \cos x = \sqrt{17} \cos(x - 1.326).$$

The maximum value of $\sqrt{17}$ occurs when:

$$\cos(x - 1.326) = 1$$

$$x - 1.326 = \cos^{-1}(1)$$

$$x - 1.326 = 0$$

$$x = 1.326 \text{ (to 3 d.p.)}$$

$$k = \sqrt{1^2 + 4^2}$$

$$= \sqrt{17}$$

$$\tan a = \frac{k \sin a}{k \cos a} = 4$$

So:

$$a = \tan^{-1}(4)$$

$$= 1.326 \text{ (to 3 d.p.)}$$

The minimum value of $-\sqrt{17}$ occurs when:

$$\cos(x - 1.326) = -1$$

$$x - 1.326 = \cos^{-1}(-1)$$

$$x - 1.326 = \pi$$

$$x = 4.468 \text{ (to 3 d.p.)}$$

5 Solving Equations

The method of writing two trigonometric terms as one can be used to help solve equations involving both a $\sin(nx)$ and a $\cos(nx)$ term.

EXAMPLES



1. Solve $5 \cos x^\circ + \sin x^\circ = 2$ where $0 \leq x \leq 360$.

First, we write $5 \cos x^\circ + \sin x^\circ$ in the form $k \cos(x^\circ - a^\circ)$:

$$\begin{aligned} 5 \cos x^\circ + \sin x^\circ &= k \cos(x^\circ - a^\circ) \\ &= k \cos x^\circ \cos a^\circ + k \sin x^\circ \sin a^\circ \\ &= (k \cos a^\circ) \cos x^\circ + (k \sin a^\circ) \sin x^\circ \end{aligned}$$

$$k \cos a^\circ = 5 \qquad k = \sqrt{5^2 + 1^2} \qquad \tan a^\circ = \frac{k \sin a^\circ}{k \cos a^\circ} = \frac{1}{5}$$

$$k \sin a^\circ = 1 \qquad = \sqrt{26} \qquad \text{So:}$$

$$\begin{array}{c|c} 180^\circ - a^\circ & \begin{array}{c} \checkmark S \\ \hline \checkmark A \end{array} \\ \hline 180^\circ + a^\circ & \begin{array}{c} \checkmark T \\ \hline \checkmark C \end{array} \\ \hline & \begin{array}{c} a^\circ \\ \hline 360^\circ - a^\circ \end{array} \end{array} \qquad \begin{aligned} a &= \tan^{-1}\left(\frac{1}{5}\right) \\ &= 11.3 \text{ (to 1 d.p.)} \end{aligned}$$

Hence a is in the first quadrant.

$$\text{Hence } 5 \cos x^\circ + \sin x^\circ = \sqrt{26} \cos(x^\circ - 11.3^\circ).$$

Now we use this to help solve the equation:

$$\begin{aligned} 5 \cos x^\circ + \sin x^\circ &= 2 \\ \sqrt{26} \cos(x^\circ - 11.3^\circ) &= 2 \\ \cos(x^\circ - 11.3^\circ) &= \frac{2}{\sqrt{26}} \end{aligned}$$

$$\begin{array}{c|c} 180^\circ - x^\circ & \begin{array}{c} S \\ \hline A \end{array} \\ \hline 180^\circ + x^\circ & \begin{array}{c} T \\ \hline C \end{array} \\ \hline & \begin{array}{c} x^\circ \\ \hline 360^\circ - x^\circ \end{array} \end{array} \qquad \begin{aligned} x - 11.3 &= \cos^{-1}\left(\frac{2}{\sqrt{26}}\right) \\ &= 66.9 \text{ (to 2 d.p.)} \end{aligned}$$

$$x - 11.3 = 66.9 \quad \text{or} \quad 360 - 66.9$$

$$x - 11.3 = 66.9 \quad \text{or} \quad 293.1$$

$$x = 78.2 \quad \text{or} \quad 304.4.$$



2. Solve $2 \cos 2x + 3 \sin 2x = 1$ where $0 \leq x \leq 2\pi$.

First, we write $2 \cos 2x + 3 \sin 2x$ in the form $k \cos(2x - a)$:

$$\begin{aligned} 2 \cos 2x + 3 \sin 2x &= k \cos(2x - a) \\ &= k \cos 2x \cos a + k \sin 2x \sin a \\ &= (k \cos a) \cos 2x + (k \sin a) \sin 2x \end{aligned}$$

$$k \cos a = 2 \qquad k = \sqrt{2^2 + (-3)^2} \qquad \tan a = \frac{k \sin a}{k \cos a} = \frac{3}{2}$$

$$k \sin a = 3 \qquad = \sqrt{4+9} \qquad \text{So:}$$

$$\begin{array}{c} \pi - a \quad \checkmark \text{S} \mid \text{A} \checkmark \checkmark \\ \hline \pi + a \quad \text{T} \mid \text{C} \checkmark \\ \quad \quad \quad 2\pi - a \end{array} \qquad = \sqrt{13} \qquad a = \tan^{-1}\left(\frac{3}{2}\right) \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad = 0.983 \text{ (to 3 d.p.)}$$

Hence a is in the first quadrant.

$$\text{Hence } 2 \cos 2x + 3 \sin 2x = \sqrt{13} \cos(2x - 0.983).$$

Now we use this to help solve the equation:

$$\begin{array}{c} 2 \cos 2x + 3 \sin 2x = 1 \\ \sqrt{13} \cos(2x - 0.983) = 1 \\ \cos(2x - 0.983) = \frac{1}{\sqrt{13}} \end{array} \qquad \begin{array}{c} \pi - 2x \quad \text{S} \mid \text{A} \checkmark \\ \hline \pi + 2x \quad \text{T} \mid \text{C} \checkmark \\ \quad \quad \quad 2\pi - 2x \end{array} \qquad \begin{array}{c} 0 < x < 2\pi \\ 0 < 2x < 4\pi \end{array}$$

$$\begin{aligned} 2x - 0.983 &= \cos^{-1}\left(\frac{1}{\sqrt{13}}\right) \\ &= 1.290 \text{ (to 3 d.p.)} \end{aligned}$$

$$2x - 0.983 = 1.290 \text{ or } 2\pi - 1.290$$

$$\text{or } 2\pi + 1.290 \text{ or } 2\pi + 2\pi - 1.290$$

$$\text{or } \underline{2\pi + 2\pi + 1.290}$$

$$2x - 0.983 = 1.290 \text{ or } 4.993 \text{ or } 7.573 \text{ or } 11.276$$

$$2x = 2.273 \text{ or } 5.976 \text{ or } 8.556 \text{ or } 12.259$$

$$x = 1.137 \text{ or } 2.988 \text{ or } 4.278 \text{ or } 6.130$$

6 Sketching Graphs of $y = p \cos x + q \sin x$

Expressing $p \cos x + q \sin x$ in the form $k \cos(x - a)$ enables us to sketch the graph of $y = p \cos x + q \sin x$.

EXAMPLES

1. (a) Write $7 \cos x^\circ + 6 \sin x^\circ$ in the form $k \cos(x^\circ - a^\circ)$, $0 \leq a \leq 360$.



(b) Hence sketch the graph of $y = 7 \cos x^\circ + 6 \sin x^\circ$ for $0 \leq x \leq 360$.

(a) First, we write $7 \cos x^\circ + 6 \sin x^\circ$ in the form $k \cos(x^\circ - a^\circ)$:

$$\begin{aligned} 7 \cos x^\circ + 6 \sin x^\circ &= k \cos(x^\circ - a^\circ) \\ &= k \cos x^\circ \cos a^\circ + k \sin x^\circ \sin a^\circ \\ &= (k \cos a^\circ) \cos x^\circ + (k \sin a^\circ) \sin x^\circ \end{aligned}$$

$$k \cos a^\circ = 7 \qquad k = \sqrt{6^2 + 7^2} \qquad \tan a^\circ = \frac{k \sin a^\circ}{k \cos a^\circ} = \frac{6}{7}$$

$$k \sin a^\circ = 6 \qquad = \sqrt{36 + 49}$$

So:

$$\begin{array}{c|c} 180^\circ - a^\circ & \text{S} \quad \text{A} \\ \hline 180^\circ + a^\circ & \text{T} \quad \text{C} \end{array} \begin{array}{c} \checkmark \\ \checkmark \\ \checkmark \\ \checkmark \end{array}$$

$$= \sqrt{85}$$

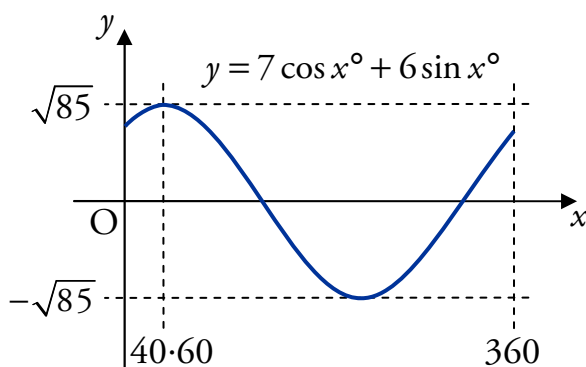
$$a = \tan^{-1}\left(\frac{6}{7}\right)$$

$$= 40.6 \text{ (to 1 d.p.)}$$

Hence a is in the first quadrant.

$$\text{Hence } 7 \cos x^\circ + 6 \sin x^\circ = \sqrt{85} \cos(x^\circ - 40.6^\circ).$$

(b) Now we can sketch the graph of $y = 7 \cos x^\circ + 6 \sin x^\circ$:



2. Sketch the graph of $y = \sin x^\circ + \sqrt{3} \cos x^\circ$ for $0 \leq x \leq 360$.

First, we write $\sin x^\circ + \sqrt{3} \cos x^\circ$ in the form $k \cos(x^\circ - a^\circ)$:

$$\begin{aligned} \sin x^\circ + \sqrt{3} \cos x^\circ &= k \cos(x^\circ - a^\circ) \\ &= k \cos x^\circ \cos a^\circ + k \sin x^\circ \sin a^\circ \\ &= (k \cos a^\circ) \cos x^\circ + (k \sin a^\circ) \sin x^\circ \end{aligned}$$

$$k \cos a^\circ = \sqrt{3} \qquad k = \sqrt{1^2 + \sqrt{3}^2} \qquad \tan a^\circ = \frac{k \sin a^\circ}{k \cos a^\circ} = \frac{1}{\sqrt{3}}$$

$$k \sin a^\circ = 1 \qquad = \sqrt{1+3} \qquad \text{So:}$$

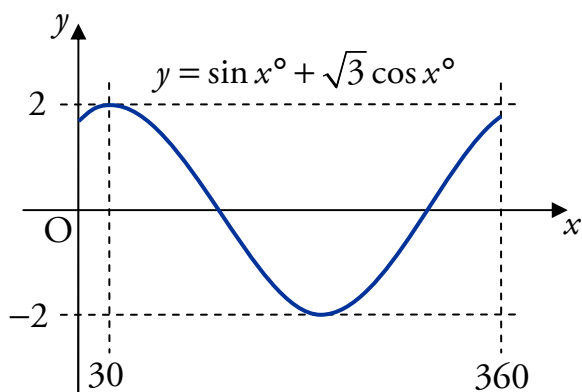
$$180^\circ - a^\circ \quad \begin{array}{c|c} \sqrt{S} & A \sqrt{\checkmark} \\ \hline 180^\circ + a^\circ & T \quad C \sqrt{\checkmark} \end{array} \quad 360^\circ - a^\circ \qquad a = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\qquad \qquad \qquad = 2 \qquad \qquad \qquad = 30$$

Hence a is in the first quadrant.

Hence $\sin x^\circ + \sqrt{3} \cos x^\circ = 2 \cos(x^\circ - 30^\circ)$.

Now we can sketch the graph of $y = \sin x^\circ + \sqrt{3} \cos x^\circ$:



3. (a) Write $5 \sin x^\circ - \sqrt{11} \cos x^\circ$ in the form $k \sin(x^\circ - a^\circ)$, $0 \leq a \leq 360$.



(b) Hence sketch the graph of $y = 5 \sin x^\circ - \sqrt{11} \cos x^\circ + 2$, $0 \leq x \leq 360$.

$$(a) \quad 5 \sin x^\circ - \sqrt{11} \cos x^\circ = k \sin(x^\circ - a^\circ)$$

$$= k \sin x^\circ \cos a^\circ + k \cos x^\circ \sin a^\circ$$

$$= (k \cos a^\circ) \sin x^\circ + (k \sin a^\circ) \cos x^\circ$$

$$k \cos a^\circ = 5$$

$$k \sin a^\circ = \sqrt{11}$$

$$180^\circ - a^\circ \quad \begin{array}{c} \checkmark S \\ \hline \checkmark A \end{array} \begin{array}{c} | \\ \hline \checkmark \end{array} \begin{array}{c} a^\circ \\ \checkmark \end{array}$$

$$180^\circ + a^\circ \quad \begin{array}{c} \checkmark T \\ \hline \checkmark C \end{array} \begin{array}{c} | \\ \hline \checkmark \end{array} \begin{array}{c} 360^\circ - a^\circ \\ \checkmark \end{array}$$

$$k = \sqrt{5^2 + \sqrt{11}^2}$$

$$= \sqrt{25 + 11}$$

$$= \sqrt{36}$$

$$= 6$$

$$\tan a^\circ = \frac{k \sin a^\circ}{k \cos a^\circ} = \frac{\sqrt{11}}{5}$$

So:

$$a = \tan^{-1}\left(\frac{\sqrt{11}}{5}\right)$$

$$= 33.6 \text{ (to 1 d.p.)}$$

Hence a is in the first quadrant.

$$\text{Hence } 5 \sin x^\circ - \sqrt{11} \cos x^\circ = 6 \sin(x^\circ - 33.6^\circ).$$

(b) Now sketch the graph of

$$y = 5 \sin x^\circ - \sqrt{11} \cos x^\circ + 2 = 6 \sin(x^\circ - 33.6^\circ) + 2:$$

