Higher

Mathematics

UNIT 2 OUTCOME 3

Trigonometry

Contents

HSN22300

OUTCOME 3

Trigonometry

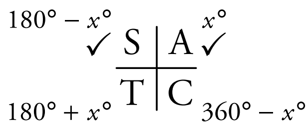
1 Solving Trigonometric Equations

You should already be familiar with solving some trigonometric equations.

EXAMPLES

1. Solve $\sin x° = \frac{1}{2}$ for $0 < x < 360$.

$\sin x° = \frac{1}{2}$



Since $\sin x°$ is positive

First quadrant solution:

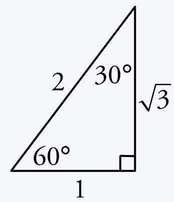
$$x = \sin^{-1}\left(\tfrac{1}{2}\right)$$
$$= 30.$$

$x = 30$ or $180 - 30$

$x = 30$ or 150.

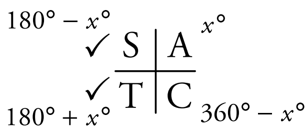
Remember

The exact value triangle:



2. Solve $\cos x° = -\frac{1}{\sqrt{5}}$ for $0 < x < 360$.

$\cos x° = -\frac{1}{\sqrt{5}}$



Since $\cos x°$ is negative

$$x = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$
$$= 63{\cdot}435 \quad \text{(to 3 d.p.)}.$$

$x = 180 - 63{\cdot}435$ or $180 + 63{\cdot}435$

$x = 116{\cdot}565$ or $243{\cdot}435$.

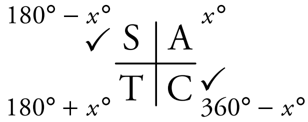
3. Solve $\sin x° = 3$ for $0 < x < 360$.

There are no solutions since $-1 \leq \sin x° \leq 1$.

Note that $-1 \leq \cos x° \leq 1$, so $\cos x° = 3$ also has no solutions.

4. Solve $\tan x° = -5$ for $0 < x < 360$.

$\tan x° = -5$



Since $\tan x°$ is negative

$$x = \tan^{-1}(5)$$
$$= 78{\cdot}690 \quad \text{(to 3 d.p.)}.$$

$x = 180 - 78{\cdot}690 \quad \text{or} \quad 360 - 78{\cdot}690$

$x = 101{\cdot}310 \quad \text{or} \quad 281{\cdot}310.$

Note

All trigonometric equations we will meet can be reduced to problems like those above. The only differences are:

- the solutions could be required in radians – in this case, the question will not have a degree symbol, e.g. "Solve $3\tan x = 1$" rather than "$3\tan x° = 1$";
- exact value solutions could be required in the non-calculator paper – you will be expected to know the exact values for 0, 30, 45, 60 and 90 degrees.

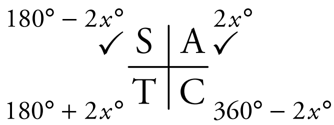
Questions can be worked through in degrees or radians, but make sure the final answer is given in the units asked for in the question.

EXAMPLES

5. Solve $2\sin 2x° - 1 = 0$ where $0 \leq x \leq 360$.

$2\sin 2x° = 1$

$\sin 2x° = \frac{1}{2}$



$0 \leq x \leq 360$

$0 \leq 2x \leq 720$

$$2x = \sin^{-1}\left(\tfrac{1}{2}\right)$$
$$= 30.$$

$2x = 30 \quad \text{or} \quad 180 - 30$

$\quad \text{or} \quad 360 + 30 \quad \text{or} \quad 360 + 180 - 30$

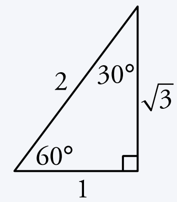
$\quad \text{or} \quad$ ~~$360 + 360 + 30$~~

$2x = 30 \quad \text{or} \quad 150 \quad \text{or} \quad 390 \quad \text{or} \quad 510$

$x = 15 \quad \text{or} \quad 75 \quad \text{or} \quad 195 \quad \text{or} \quad 255.$

Remember

The exact value triangle:



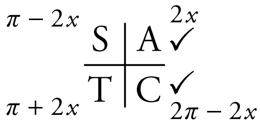
Note

There are more solutions every 360°, since $\sin(30°) = \sin(30° + 360°) = \ldots$

So keep adding 360 until $2x > 720$.

6. Solve $\sqrt{2}\cos 2x = 1$ where $0 \le x \le \pi$.

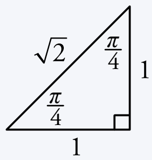
$\cos 2x = \frac{1}{\sqrt{2}}$



$0 \le x \le \pi$

$0 \le 2x \le 2\pi$

Remember
The exact value triangle:



$2x = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{\pi}{4}$.

$2x = \frac{\pi}{4}$ or $2\pi - \frac{\pi}{4}$

or ~~$2\pi + \frac{\pi}{4}$~~

$2x = \frac{\pi}{4}$ or $\frac{7\pi}{4}$

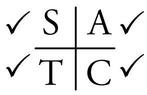
$x = \frac{\pi}{8}$ or $\frac{7\pi}{8}$.

7. Solve $4\cos^2 x = 3$ where $0 < x < 2\pi$.

$(\cos x)^2 = \frac{3}{4}$

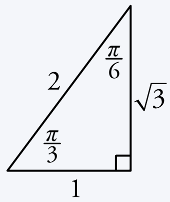
$\cos x = \pm\sqrt{\frac{3}{4}}$

$\cos x = \pm\frac{\sqrt{3}}{2}$



Since $\cos x$ can be positive or negative

Remember
The exact value triangle:



$x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{6}$.

$x = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6}$ or $\pi + \frac{\pi}{6}$ or $2\pi - \frac{\pi}{6}$

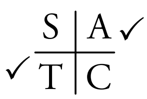
or ~~$2\pi + \frac{\pi}{6}$~~

$x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $\frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

8. Solve $3\tan(3x° - 20°) = 5$ where $0 \le x \le 360$.

$3\tan(3x° - 20°) = 5$

$\tan(3x° - 20°) = \frac{5}{3}$



$0 \le x \le 360$

$0 \le 3x \le 1080$

$-20 \le 3x - 20 \le 1060$

$3x - 20 = \tan^{-1}\left(\frac{5}{3}\right)$

$= 59{\cdot}036$ (to 3 d.p.).

$3x - 20 = 59{\cdot}036$ or $180 + 59{\cdot}036$

or $360 + 59{\cdot}036$ or $360 + 180 + 59{\cdot}036$

or $360 + 360 + 59{\cdot}036$ or $360 + 360 + 180 + 59{\cdot}036$

or ~~$360 + 360 + 360 + 59{\cdot}036$~~

$$3x - 20 = 59{\cdot}036 \text{ or } 239{\cdot}036 \text{ or } 419{\cdot}036$$
$$\text{or } 599{\cdot}036 \text{ or } 779{\cdot}036 \text{ or } 959{\cdot}036$$
$$3x = 79{\cdot}036 \text{ or } 259{\cdot}036 \text{ or } 439{\cdot}036$$
$$\text{or } 619{\cdot}036 \text{ or } 799{\cdot}036 \text{ or } 979{\cdot}036$$
$$x = 26{\cdot}35 \text{ or } 86{\cdot}35 \text{ or } 146{\cdot}35 \text{ or } 206{\cdot}35 \text{ or } 266{\cdot}35 \text{ or } 326{\cdot}35.$$

9. Solve $\cos\left(2x + \frac{\pi}{3}\right) = 0{\cdot}812$ for $0 < x < 2\pi$.

$$\cos\left(2x + \tfrac{\pi}{3}\right) = 0{\cdot}812$$

S	A ✓
T	C ✓

$$0 < x < 2\pi$$
$$0 < 2x < 4\pi$$
$$\tfrac{\pi}{3} < 2x + \tfrac{\pi}{3} < 4\pi + \tfrac{\pi}{3}$$
$$1{\cdot}047 < 2x + \tfrac{\pi}{3} < 13{\cdot}614 \quad \text{(to 3 d.p.)}$$

$$2x + \tfrac{\pi}{3} = \cos^{-1}(0{\cdot}812)$$
$$= 0{\cdot}623 \quad \text{(to 3 d.p.).}$$

Remember
Make sure your calculator uses radians.

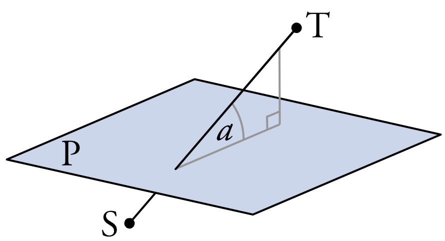
$$2x + \tfrac{\pi}{3} = \cancel{0{\cdot}623} \text{ or } 2\pi - 0{\cdot}623$$
$$\text{or } 2\pi + 0{\cdot}623 \text{ or } 2\pi + 2\pi - 0{\cdot}623$$
$$\text{or } 2\pi + 2\pi + 0{\cdot}623 \text{ or } \cancel{2\pi + 2\pi + 2\pi - 0{\cdot}623}$$
$$2x + \tfrac{\pi}{3} = 5{\cdot}660 \text{ or } 6{\cdot}906 \text{ or } 11{\cdot}943 \text{ or } 13{\cdot}189$$
$$2x = 4{\cdot}613 \text{ or } 5{\cdot}859 \text{ or } 10{\cdot}896 \text{ or } 12{\cdot}142$$
$$x = 2{\cdot}307 \text{ or } 2{\cdot}930 \text{ or } 5{\cdot}448 \text{ or } 6{\cdot}071.$$

2 Trigonometry in Three Dimensions

It is possible to solve trigonometric problems in three dimensions using techniques we already know from two dimensions. The use of sketches is often helpful.

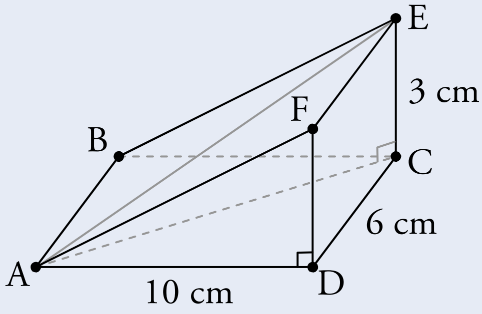
The angle between a line and a plane

The angle a between the plane P and the line ST is calculated by adding a line perpendicular to the plane and then using basic trigonometry.



EXAMPLE

1. The triangular prism ABCDEF is shown below.

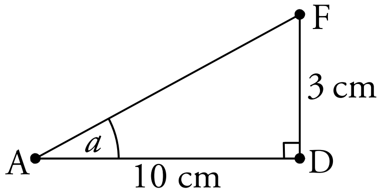


Calculate the acute angle between:

(a) The line AF and the plane ABCD.

(b) AE and ABCD.

(a) Start with a sketch:



$$\tan a = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{3}{10}$$

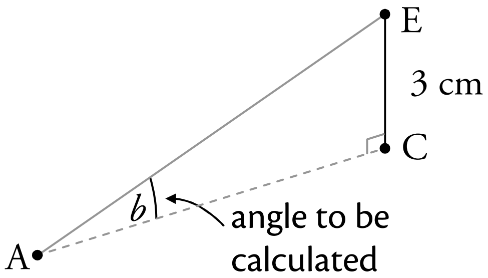
$$a = \tan^{-1}\left(\tfrac{3}{10}\right)$$

$$= 16{\cdot}699° \quad (\text{or } 0{\cdot}291 \text{ radians}) \quad (\text{to 3 d.p.}).$$

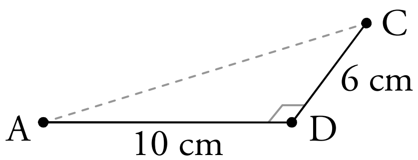
Note

Since the angle is in a right-angled triangle, it must be acute so there is no need for a CAST diagram.

(b) Again, make a sketch:



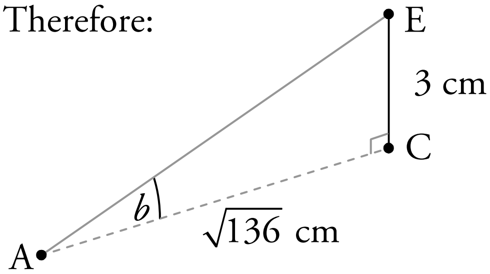
We need to calculate the length of AC first using Pythagoras's Theorem:



$$AC = \sqrt{10^2 + 6^2}$$

$$= \sqrt{136}.$$

Therefore:



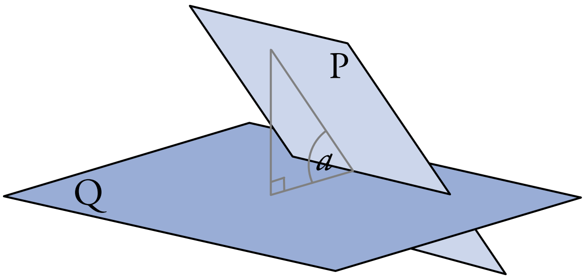
$$\tan b = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{3}{\sqrt{136}}$$

$$b = \tan^{-1}\left(\frac{3}{\sqrt{136}}\right)$$

$$= 14{\cdot}426° \quad (\text{or } 0{\cdot}252 \text{ radians}) \quad (\text{to 3 d.p.}).$$

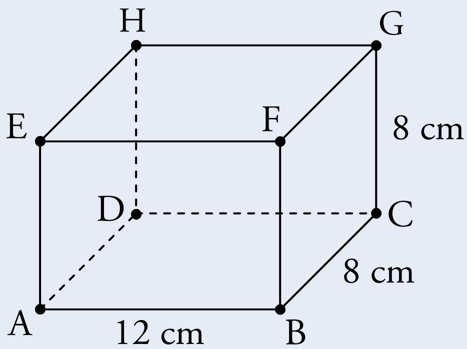
The angle between two planes

The angle a between planes P and Q is calculated by adding a line perpendicular to Q and then using basic trigonometry.



EXAMPLE

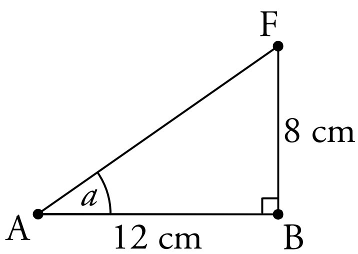
2. ABCDEFGH is a cuboid with dimensions $12 \times 8 \times 8$ cm as shown below.



(a) Calculate the size of the angle between the planes AFGD and ABCD.

(b) Calculate the size of the acute angle between the diagonal planes AFGD and BCHE.

(a) Start with a sketch:



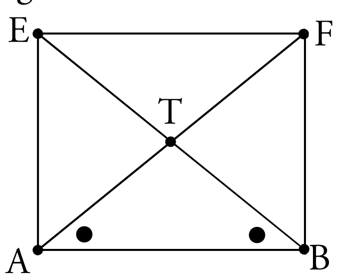
$$\tan a = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{8}{12}$$

$$a = \tan^{-1}\left(\tfrac{2}{3}\right)$$

$$= 33{\cdot}690° \quad (\text{or } 0{\cdot}588 \text{ radians}) \quad (\text{to 3 d.p.}).$$

Note
Angle GDC is the same size as angle FAB.

(b) Again, make a sketch:



Let AF and BE intersect at T.

$\triangle ATB$ is isosceles, so $T\hat{A}B = T\hat{B}A = 33{\cdot}690°$.

$$A\hat{T}B = 180° - (33{\cdot}690° + 33{\cdot}690°)$$

$$= 112{\cdot}620°.$$

So the acute angle is:

$$B\hat{T}F = A\hat{T}E = 180° - 112{\cdot}620°$$

$$= 67{\cdot}380° \quad (\text{or } 1{\cdot}176 \text{ radians}) \quad (\text{to 3 d.p.}).$$

Note
The angle could also have been calculated using rectangle DCGH.

3 Compound Angles

When we add or subtract angles, the result is called a **compound angle**.

For example, $45°+30°$ is a compound angle. Using a calculator, we find:

- $\sin(45°+30°)=\sin(75°)=0{\cdot}966$;
- $\sin(45°)+\sin(30°)=1{\cdot}207$ (both to 3 d.p.).

This shows that $\sin(A+B)$ is *not* equal to $\sin A+\sin B$. Instead, we can use the following identities:

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$
$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B.$$

These are given in the exam in a condensed form:

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$
$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B.$$

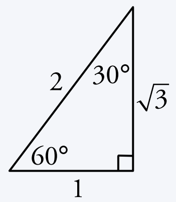
EXAMPLES

1. Expand and simplify $\cos(x°+60°)$.

$$\cos(x°+60°)=\cos x°\cos 60°-\sin x°\sin 60°$$
$$=\tfrac{1}{2}\cos x°-\tfrac{\sqrt{3}}{2}\sin x°.$$

Remember

The exact value triangle:



2. Show that $\sin(a+b)=\sin a\cos b+\cos a\sin b$ for $a=\frac{\pi}{6}$ and $b=\frac{\pi}{3}$.

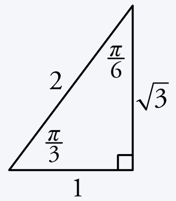
$$\text{LHS}=\sin(a+b)=\sin\left(\tfrac{\pi}{6}+\tfrac{\pi}{3}\right)=\sin\left(\tfrac{\pi}{2}\right)=1.$$

$$\text{RHS}=\sin a\cos b+\cos a\sin b=\sin\tfrac{\pi}{6}\cos\tfrac{\pi}{3}+\cos\tfrac{\pi}{6}\sin\tfrac{\pi}{3}=\left(\tfrac{1}{2}\times\tfrac{1}{2}\right)+\left(\tfrac{\sqrt{3}}{2}\times\tfrac{\sqrt{3}}{2}\right)=\tfrac{1}{4}+\tfrac{3}{4}=1.$$

Since LHS = RHS, the claim is true for $a=\frac{\pi}{6}$ and $b=\frac{\pi}{3}$.

Remember

The exact value triangle:

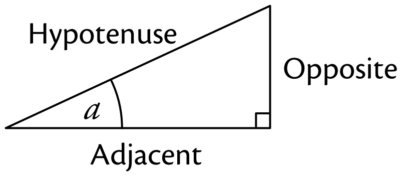


3. Find the exact value of $\sin 75°$.

$$\begin{aligned}\sin 75° &= \sin(45° + 30°)\\ &= \sin 45° \cos 30° + \cos 45° \sin 30°\\ &= \left(\tfrac{1}{\sqrt{2}} \times \tfrac{\sqrt{3}}{2}\right) + \left(\tfrac{1}{\sqrt{2}} \times \tfrac{1}{2}\right)\\ &= \tfrac{\sqrt{3}}{2\sqrt{2}} + \tfrac{1}{2\sqrt{2}}\\ &= \tfrac{\sqrt{3}+1}{2\sqrt{2}}\\ &= \tfrac{\sqrt{6}+\sqrt{2}}{4}.\end{aligned}$$

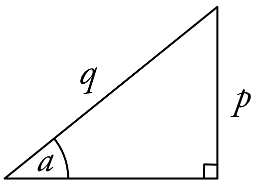
Finding Trigonometric Ratios

You should already be familiar with the following formulae (SOH CAH TOA).



$$\sin a = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \cos a = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \tan a = \frac{\text{Opposite}}{\text{Adjacent}}.$$

If we have $\sin a = \frac{p}{q}$ where $0 < a < \frac{\pi}{2}$, then we can form a right-angled triangle to represent this ratio.



Since $\sin a = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{p}{q}$ then:

- the side opposite a has length p;
- the hypotenuse has length q.

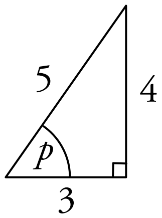
The length of the unknown side can be found using Pythagoras's Theorem.

Once the length of each side is known, we can find $\cos a$ and $\tan a$ using SOH CAH TOA.

The method is similar if we know $\cos a$ and want to find $\sin a$ or $\tan a$.

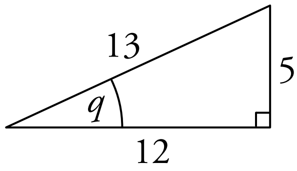
EXAMPLES

4. Acute angles p and q are such that $\sin p = \frac{4}{5}$ and $\sin q = \frac{5}{13}$. Show that $\sin(p+q) = \frac{63}{65}$.



$\sin p = \frac{4}{5}$

$\cos p = \frac{3}{5}$



$\sin q = \frac{5}{13}$

$\cos q = \frac{12}{13}$

$$\begin{aligned}\sin(p+q) &= \sin p\cos q + \cos p\sin q\\ &= \left(\tfrac{4}{5}\times\tfrac{12}{13}\right) + \left(\tfrac{3}{5}\times\tfrac{5}{13}\right)\\ &= \tfrac{48}{65} + \tfrac{15}{65}\\ &= \tfrac{63}{65}.\end{aligned}$$

Note
Since "Show that" is used in the question, all of this working is required.

Confirming Identities

EXAMPLES

5. Show that $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$.

$$\begin{aligned}&\sin\left(x-\tfrac{\pi}{2}\right)\\ &= \sin x\cos\tfrac{\pi}{2} - \cos x\sin\tfrac{\pi}{2}\\ &= \sin x \times 0 - \cos x \times 1\\ &= -\cos x.\end{aligned}$$

6. Show that $\dfrac{\sin(s+t)}{\cos s\cos t} = \tan s + \tan t$ for $\cos s \neq 0$ and $\cos t \neq 0$.

$$\begin{aligned}\frac{\sin(s+t)}{\cos s\cos t} &= \frac{\sin s\cos t + \cos s\sin t}{\cos s\cos t}\\ &= \frac{\sin s\cos t}{\cos s\cos t} + \frac{\cos s\sin t}{\cos s\cos t}\\ &= \frac{\sin s}{\cos s} + \frac{\sin t}{\cos t}\\ &= \tan s + \tan t.\end{aligned}$$

Remember
$\dfrac{\sin x}{\cos x} = \tan x.$

4 Double-Angle Formulae

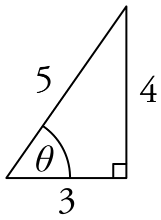
Using the compound angle identities with $A = B$, we obtain expressions for $\sin 2A$ and $\cos 2A$. These are called **double-angle formulae**.

$$\sin 2A = 2\sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A.$$

Note that these are given in the exam.

EXAMPLES

1. Given that $\tan\theta = \frac{4}{3}$, where $0 < \theta < \frac{\pi}{2}$, find the exact value of $\sin 2\theta$ and $\cos 2\theta$.



$\sin\theta = \frac{4}{3}$

$\cos\theta = \frac{3}{5}$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2 \times \tfrac{4}{5} \times \tfrac{3}{5}$$
$$= \tfrac{24}{25}.$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \left(\tfrac{3}{5}\right)^2 - \left(\tfrac{4}{5}\right)^2$$
$$= \tfrac{9}{25} - \tfrac{16}{25}$$
$$= -\tfrac{7}{25}.$$

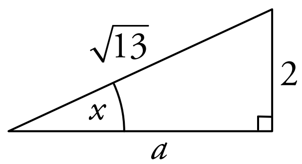
Note
Any of the $\cos 2A$ formulae could have been used here.

2. Given that $\cos 2x = \frac{5}{13}$, where $0 < x < \pi$, find the exact values of $\sin x$ and $\cos x$.

Since $\cos 2x = 1 - 2\sin^2 x$,

$$1 - 2\sin^2 x = \tfrac{5}{13}$$
$$2\sin^2 x = \tfrac{8}{13}$$
$$\sin^2 x = \tfrac{8}{26}$$
$$= \tfrac{4}{13}$$
$$\sin x = \pm\tfrac{2}{\sqrt{13}}.$$

We are told that $0 < x < \pi$, so only $\sin x = \frac{2}{\sqrt{13}}$ is possible.



$$a = \sqrt{\sqrt{13}^2 - 2^2} = \sqrt{13 - 4} = \sqrt{9} = 3.$$

So $\cos x = \frac{3}{\sqrt{13}}$.

5 Further Trigonometric Equations

We will now consider trigonometric equations where double-angle formulae can be used to find solutions. These equations will involve:

- $\sin 2x$ and either $\sin x$ or $\cos x$;
- $\cos 2x$ and $\cos x$;
- $\cos 2x$ and $\sin x$.

Remember
The double-angle formulae are given in the exam.

Solving equations involving sin2*x* and either sin*x* or cos*x*

EXAMPLE

1. Solve $\sin 2x° = -\sin x°$ for $0 \leq x < 360$.

$$2\sin x° \cos x° = -\sin x°$$
$$2\sin x° \cos x° + \sin x° = 0$$
$$\sin x°(2\cos x° + 1) = 0$$

- Replace $\sin 2x$ using the double angle formula
- Take all terms to one side, making the equation equal to zero
- Factorise the expression and solve

$\sin x° = 0$

$x = 0$ or 180 or ~~360~~

$2\cos x° + 1 = 0$

$\cos x° = -\frac{1}{2}$

$x = 180 - 60$ or $180 + 60$

$= 120$ or 240.

✓ S | A
✓ T | C

$x = \cos^{-1}\left(\frac{1}{2}\right)$

$= 60$

So $x = 0$ or 120 or 180 or 240.

Solving equations involving cos2*x* and cos*x*

EXAMPLE

2. Solve $\cos 2x = \cos x$ for $0 \leq x \leq 2\pi$.

$$\cos 2x = \cos x$$
$$2\cos^2 x - 1 = \cos x$$
$$2\cos^2 x - \cos x - 1 = 0$$
$$(2\cos x + 1)(\cos x - 1) = 0$$

- Replace $\cos 2x$ by $2\cos^2 x - 1$
- Take all terms to one side, making a quadratic equation in $\cos x$
- Solve the quadratic equation (using factorisation or the quadratic formula)

$2\cos x + 1 = 0$

$\cos x = -\frac{1}{2}$

$x = \pi - \frac{\pi}{3}$ or $\pi + \frac{\pi}{3}$

$= \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

✓ S | A
✓ T | C

$x = \cos^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{3}$

$\cos x - 1 = 0$

$\cos x = 1$

$x = 0$ or 2π.

So $x = 0$ or $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ or 2π.

Solving equations involving cos2*x* and sin*x*

EXAMPLE

3. Solve $\cos 2x = \sin x$ for $0 < x < 2\pi$.

$$\begin{aligned} \cos 2x &= \sin x \\ 1 - 2\sin^2 x &= \sin x \\ 2\sin^2 x + \sin x - 1 &= 0 \\ (2\sin x - 1)(\sin x + 1) &= 0 \end{aligned}$$

- Replace $\cos 2x$ by $1 - 2\sin^2 x$
- Take all terms to one side, making a quadratic equation in $\sin x$
- Solve the quadratic equation (using factorisation or the quadratic formula)

$$\begin{aligned} 2\sin x - 1 &= 0 \\ \sin x &= \tfrac{1}{2} \\ x &= \tfrac{\pi}{6} \text{ or } \pi - \tfrac{\pi}{6} \\ &= \tfrac{\pi}{6} \text{ or } \tfrac{5\pi}{6} \end{aligned}$$

✓ S	A ✓
T	C

$$\begin{aligned} x &= \sin^{-1}\left(\tfrac{1}{2}\right) \\ &= \tfrac{\pi}{6} \end{aligned}$$

$$\begin{aligned} \sin x + 1 &= 0 \\ \sin x &= -1 \\ x &= \tfrac{3\pi}{2}. \end{aligned}$$

So $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $\frac{3\pi}{2}$.