Higher

Mathematics

UNIT 1 OUTCOME 2

Functions and Graphs

Contents

HSN21200

OUTCOME 2

Functions and Graphs

1 Sets

In order to study functions and graphs, we use set theory. This requires some standard symbols and terms, which you should become familiar with.

A **set** is a collection of objects (usually numbers).

For example, $S = \{5, 6, 7, 8\}$ is a set (we just list the objects inside curly brackets).

We refer to the objects in a set as its **elements** (or members), e.g. 7 is an element of *S*. We can write this symbolically as $7 \in S$. It is also clear that 4 is *not* an element of *S*; we can write $4 \notin S$.

Given two sets *A* and *B*, we say *A* is a **subset** of *B* if all elements of *A* are also elements of *B*. For example, $\{6, 7, 8\}$ is a subset of *S*.

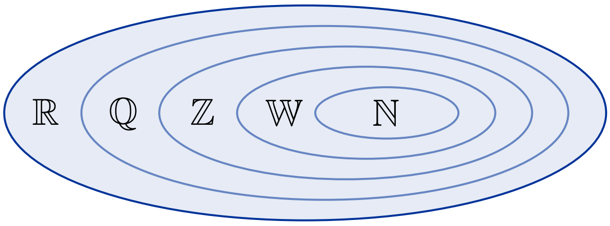
The **empty set** is the set with no elements. It is denoted by $\{\ \}$ or $\varnothing$.

Standard Sets

There are common sets of numbers which have their own symbols. Note that numbers can belong to more than one set.

$\mathbb{N}$	natural numbers	counting numbers, i.e. $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$.
$\mathbb{W}$	whole numbers	natural numbers including zero, i.e. $\mathbb{W} = \{0, 1, 2, 3, 4, \ldots\}$.
$\mathbb{Z}$	integers	positive and negative whole numbers, i.e. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
$\mathbb{Q}$	rational numbers	can be written as a fraction of integers, e.g. $-4, \frac{1}{3}, 0{\cdot}25, -\frac{1}{3}$.
$\mathbb{R}$	real numbers	all points on the number line, e.g. $-6, -\frac{1}{2}, \sqrt{2}, \frac{1}{12}, 0{\cdot}125$.

Notice that $\mathbb{N}$ is a subset of $\mathbb{W}$, which is a subset of $\mathbb{Z}$, which is a subset of $\mathbb{Q}$, which is a subset of $\mathbb{R}$. These relationships between the standard sets are illustrated in the "Venn diagram" below.



EXAMPLE

List all the numbers in the set $\mathrm{P} = \{x \in \mathbb{N} : 1 < x < 5\}$.

P contains natural numbers which are strictly greater than 1 and strictly less than 5, so:

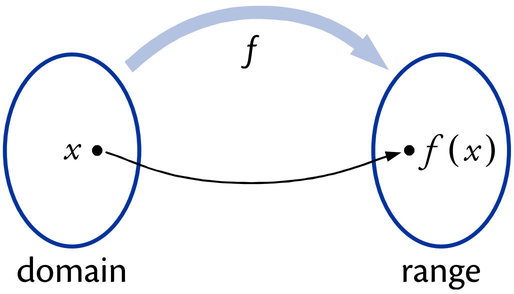
$$\mathrm{P} = \{2, 3, 4\}.$$

Note
In set notation, a colon (:) means "such that".

2 Functions

A **function** relates a set of inputs to a set of outputs, with each input related to exactly one output.

The set of inputs is called the **domain** and the resulting set of outputs is called the **range**.



A function is usually denoted by a lower case letter (e.g. f or g) and is defined using a formula of the form $f(x) = \ldots$. This specifies what the output of the function is when x is the input.

For example, if $f(x) = x^2 + 1$ then f squares the input and adds 1.

Restrictions on the Domain

The domain is the set of all possible inputs to a function, so it must be possible to evaluate the function for any element of the domain.

We are free to choose the domain, provided that the function is defined for all elements in it. If no domain is specified then we assume that it is as large as possible.

Division by Zero

It is impossible to divide by zero. So in functions involving fractions, the domain must exclude numbers which would give a denominator (bottom line) of zero.

For example, the function defined by:

$$f(x) = \frac{3}{x-5}$$

cannot have 5 in its domain, since this would make the denominator equal to zero.

The domain of f may be expressed formally as $\{x \in \mathbb{R} : x \neq 5\}$. This is read as "all x belonging to the real set such that x does not equal five".

Even Roots

Using real numbers, we cannot evaluate an even root (i.e. square root, fourth root etc.) of a negative number. So the domain of any function involving even roots must exclude numbers which would give a negative number under the root.

For example, the function defined by:

$$f(x) = \sqrt{7x-2}$$

must have $7x - 2 \geq 0$. Solving for x gives $x \geq \frac{2}{7}$, so the domain of f can be expressed formally as $\left\{x \in \mathbb{R} : x \geq \frac{2}{7}\right\}$.

EXAMPLE

1. A function g is defined by $g(x) = x - \dfrac{6}{x+4}$.

   Define a suitable domain for g.

We cannot divide by zero, so $x \neq -4$. The domain is $\{x \in \mathbb{R} : x \neq -4\}$.

Identifying the Range

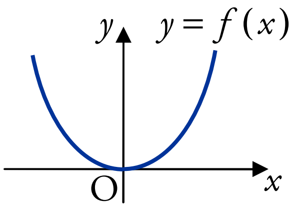
Recall that the range is the set of possible outputs. Some functions cannot produce certain values so these are not in the range.

For example:

$$f(x) = x^2$$

does not produce negative values, since any number squared is either positive or zero.

Looking at the graph of a function makes identifying its range more straightforward.



If we consider the graph of $y = f(x)$ (shown to the left) it is clear that there are no negative y-values.

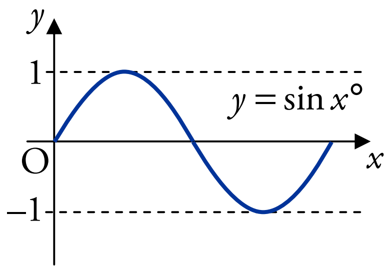
The range can be stated as $f(x) \geq 0$.

Note that the range also depends on the choice of domain. For example, if the domain of $f(x) = x^2$ is chosen to be $\{x \in \mathbb{R} : x \geq 3\}$ then the range can be stated as $f(x) \geq 9$.

EXAMPLE

2. A function f is defined by $f(x) = \sin x°$ for $x \in \mathbb{R}$. Identify its range.

Sketching the graph of $y = f(x)$ shows that $\sin x°$ only produces values from –1 to 1 inclusive.



This can be written as $-1 \leq f(x) \leq 1$.

3 Composite Functions

Two functions can be "composed" to form a new **composite function**.

For example, if we have a squaring function and a halving function, we can compose them to form a new function. We take the output from one and use it as the input for the other.

$$x \longrightarrow \boxed{\text{square}} \longrightarrow x^2 \longrightarrow \boxed{\text{halve}} \longrightarrow \frac{x^2}{2}$$

The order is important, as we get a different result in this case:

$$x \longrightarrow \boxed{\text{halve}} \longrightarrow \frac{x}{2} \longrightarrow \boxed{\text{square}} \longrightarrow \frac{x^2}{4}$$

Using function notation we have, say, $f(x)=x^2$ and $g(x)=\frac{x}{2}$.

The diagrams above show the composite functions:

$$\begin{aligned} g(f(x)) &= g\left(x^2\right) \\ &= \frac{x^2}{2} \end{aligned} \qquad\qquad \begin{aligned} f(g(x)) &= f\left(\tfrac{x}{2}\right) \\ &= \left(\tfrac{x}{2}\right)^2 = \frac{x^2}{4}. \end{aligned}$$

EXAMPLES

1. Functions f and g are defined by $f(x)=2x$ and $g(x)=x-3$. Find:
   (a) $f(2)$ (b) $f\left(g(x)\right)$ (c) $g\left(f(x)\right)$

(a) $\begin{aligned} f(2) &= 2(2) \\ &= 4. \end{aligned}$ (b) $\begin{aligned} f\left(g(x)\right) &= f(x-3) \\ &= 2(x-3). \end{aligned}$ (c) $\begin{aligned} g\left(f(x)\right) &= g(2x) \\ &= 2x-3. \end{aligned}$

2. Functions f and g are defined on suitable domains by $f(x)=x^3+1$ and $g(x)=\frac{1}{x}$.
   Find formulae for $h(x)=f\left(g(x)\right)$ and $k(x)=g\left(f(x)\right)$.

$$\begin{aligned} h(x) &= f\left(g(x)\right) \\ &= f\left(\tfrac{1}{x}\right) \\ &= \left(\tfrac{1}{x}\right)^3+1. \end{aligned} \qquad\qquad \begin{aligned} k(x) &= g\left(f(x)\right) \\ &= g\left(x^3+1\right) \\ &= \frac{1}{x^3+1}. \end{aligned}$$

4 Inverse Functions

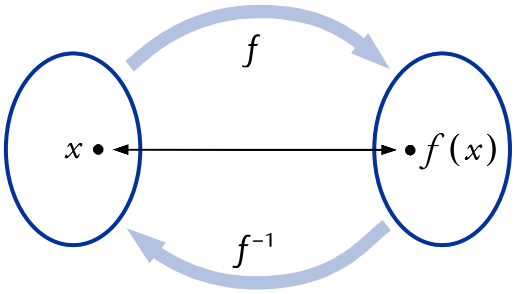
The idea of an **inverse function** is to reverse the effect of the original function. It is the “opposite” function.

You should already be familiar with this idea – for example, doubling a number can be reversed by halving the result. That is, multiplying by two and dividing by two are inverse functions.

The inverse of the function f is usually denoted f^{-1} (read as “f inverse”).

The functions f and g are said to be **inverses** if $f\left(g(x)\right) = g\left(f(x)\right) = x$.

This means that when a number is worked through a function f then its inverse f^{-1}, the result is the same as the input.



For example, $f(x) = 4x - 1$ and $g(x) = \dfrac{x+1}{4}$ are inverse functions since:

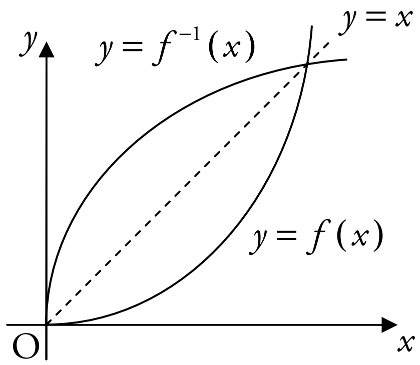
$$\begin{aligned} f\left(g(x)\right) &= f\left(\frac{x+1}{4}\right) \\ &= 4\left(\frac{x+1}{4}\right) - 1 \\ &= x + 1 - 1 \\ &= x \end{aligned}$$

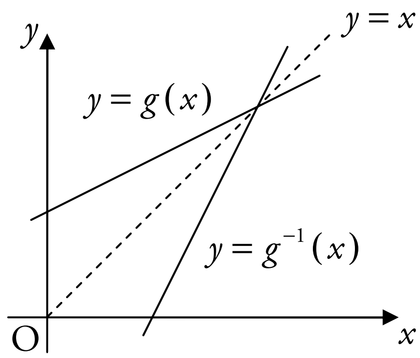
$$\begin{aligned} g\left(f(x)\right) &= g(4x-1) \\ &= \frac{(4x-1)+1}{4} \\ &= \frac{4x}{4} \\ &= x. \end{aligned}$$

Graphs of Inverses

If we have the graph of a function, then we can find the graph of its inverse by reflecting in the line $y = x$.

For example, the diagrams below show the graphs of two functions and their inverses.





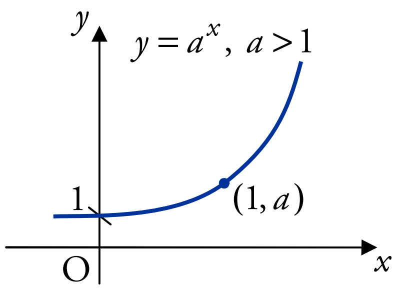
5 Exponential Functions

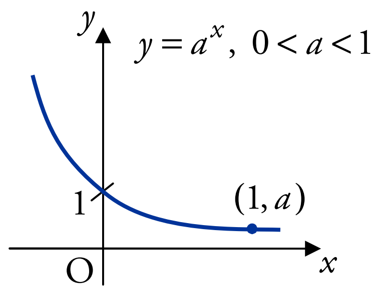
A function of the form $f(x) = a^x$ where $a, x \in \mathbb{R}$ and $a > 0$ is known as an **exponential** function to the base a.

We refer to x as the **power**, index or exponent.

Notice that when $x = 0$, $f(x) = a^0 = 1$. Also when $x = 1$, $f(x) = a^1 = a$.

Hence the graph of an exponential always passes through $(0, 1)$ and $(1, a)$:

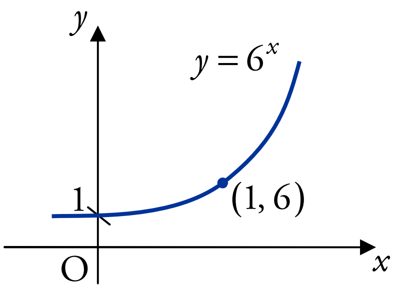




EXAMPLE

Sketch the curve with equation $y = 6^x$.

The curve passes through $(0, 1)$ and $(1, 6)$.



6 Introduction to Logarithms

Until now, we have only been able to solve problems involving exponentials when we know the index, and have to find the base. For example, we can solve $p^6 = 512$ by taking sixth roots to get $p = \sqrt[6]{512}$.

But what if we know the base and have to find the index?

To solve $6^q = 512$ for q, we need to find the power of 6 which gives 512. To save writing this each time, we use the notation $q = \log_6 512$, read as "log to the base 6 of 512". In general:

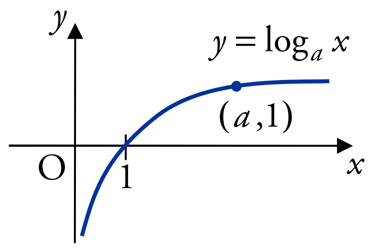
$\log_a x$ is the power of a which gives x.

The properties of logarithms will be covered in Unit 3 Outcome 3.

Logarithmic Functions

A **logarithmic** function is one in the form $f(x) = \log_a x$ where $a, x > 0$.

Logarithmic functions are inverses of exponentials, so to find the graph of $y = \log_a x$, we can reflect the graph of $y = a^x$ in the line $y = x$.

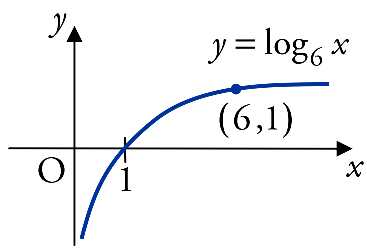


The graph of a logarithmic function always passes through $(1,0)$ and $(a,1)$.

EXAMPLE

Sketch the curve with equation $y = \log_6 x$.

The curve passes through $(1,0)$ and $(6,1)$.



7 Radians

Degrees are not the only units used to measure angles. The radian (RAD on the calculator) is an alternative measurement which is more useful in mathematics.

Degrees and radians bear the relationship:

$$\pi \text{ radians} = 180°.$$

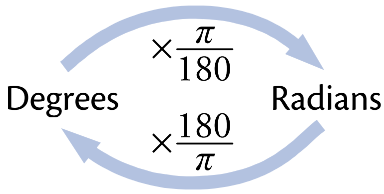
The other equivalences that you should become familiar with are:

$$30° = \tfrac{\pi}{6} \text{ radians} \qquad 45° = \tfrac{\pi}{4} \text{ radians} \qquad 60° = \tfrac{\pi}{3} \text{ radians}$$

$$90° = \tfrac{\pi}{2} \text{ radians} \qquad 135° = \tfrac{3\pi}{4} \text{ radians} \qquad 360° = 2\pi \text{ radians.}$$

Converting between degrees and radians is straightforward.

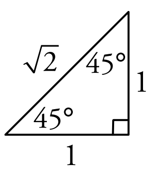
- To convert from degrees to radians, multiply by π and divide by 180.
- To convert from radians to degrees, multiply by 180 and divide by π.

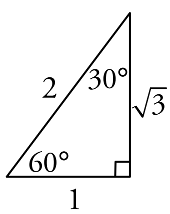


For example, $50° = 50 \times \frac{\pi}{180} = \frac{5}{18}\pi$ radians.

8 Exact Values

The following exact values must be known. You can do this by either memorising the two triangles involved, or memorising the table.





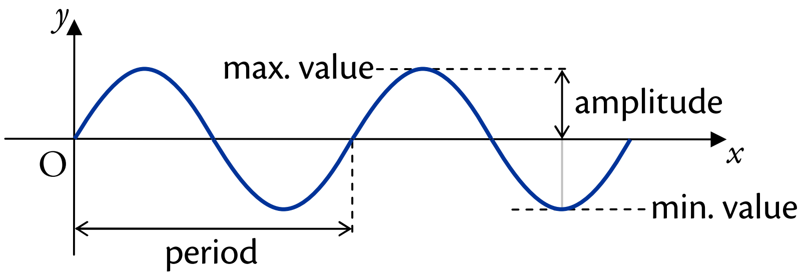
DEG	RAD	$\sin x$	$\cos x$	$\tan x$
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	–

Tip
You'll probably find it easier to remember the triangles.

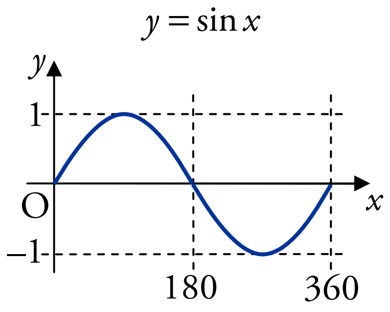
9 Trigonometric Functions

A function which has a repeating pattern in its graph is called **periodic**. The length of the smallest repeating pattern in the x-direction is called the **period**.

If the repeating pattern has a minimum and maximum value, then half of the difference between these values is called the **amplitude**.

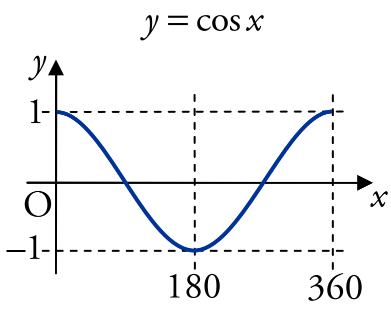


The three basic trigonometric functions (sine, cosine, and tangent) are periodic, and have graphs as shown below.



Period $= 360° = 2\pi$ radians

Amplitude $= 1$



Period $= 360° = 2\pi$ radians

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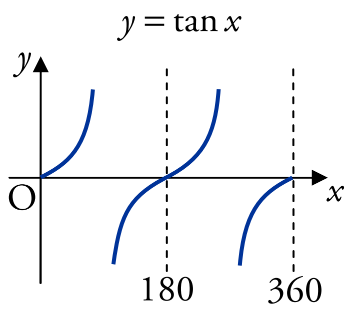
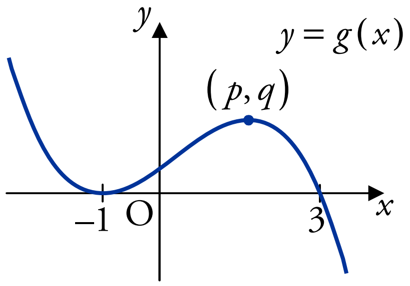


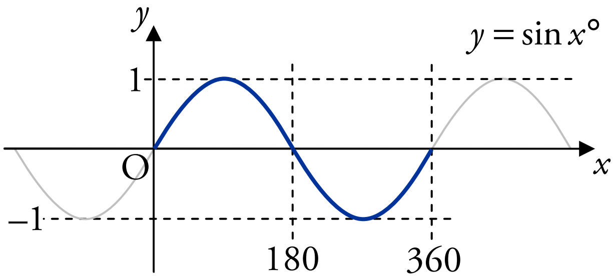
Period $= 180° = \pi$ radians

Amplitude is undefined

10 Graph Transformations

The graphs below represent two functions. One is a cubic and the other is a sine wave, focusing on the region between 0 and 360.





In the following pages we will see the effects of three different "transformations" on these graphs: translation, reflection and scaling.

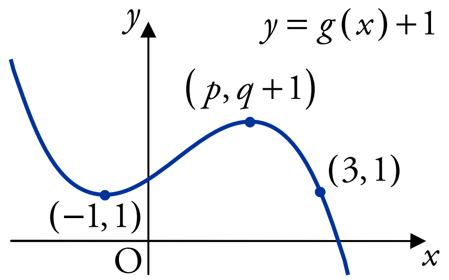
Translation

A **translation** moves every point on a graph a fixed distance in the same direction. The shape of the graph does not change.

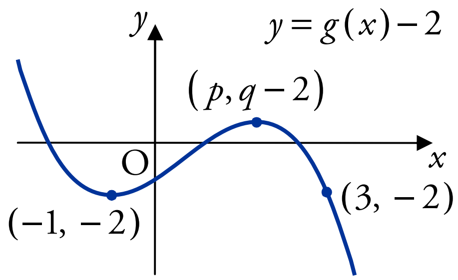
Translation parallel to the y-axis

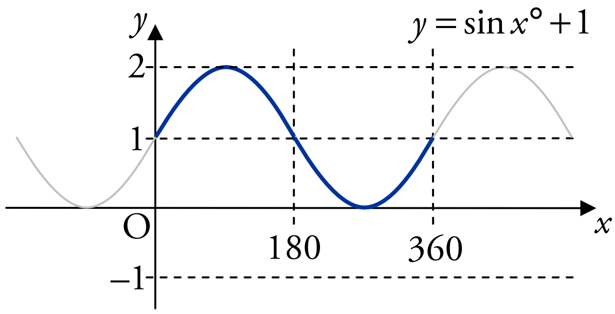
$f(x)+a$ moves the graph of $f(x)$ up or down. The graph is moved up if a is positive, and down if a is negative.

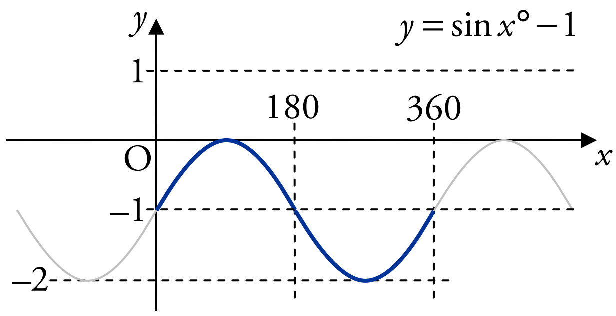
a is positive



a is negative



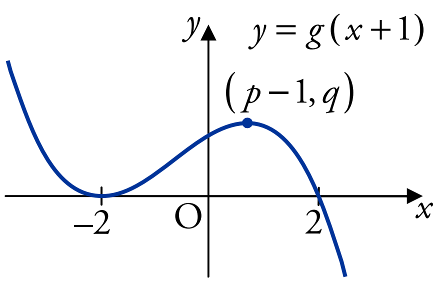




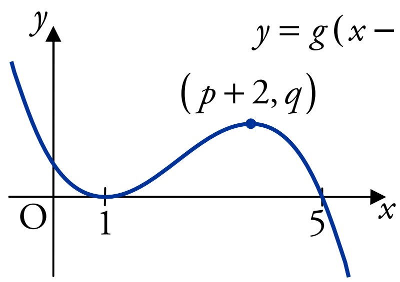
Translation parallel to the x-axis

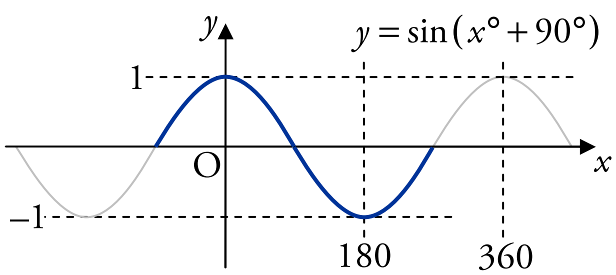
$f(x+a)$ moves the graph of $f(x)$ left or right. The graph is moved left if a is positive, and right if a is negative.

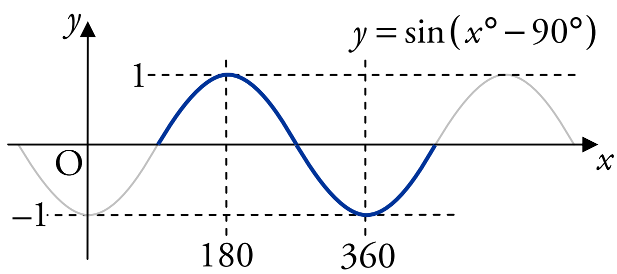
a is positive



a is negative







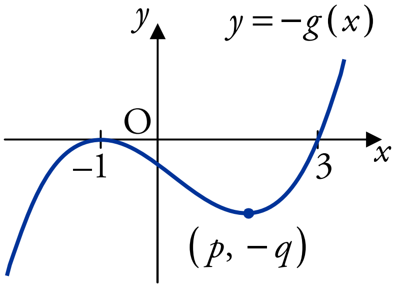
Reflection

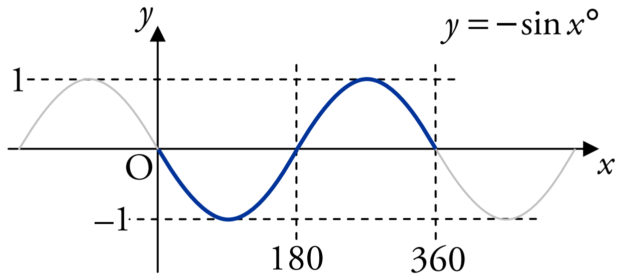
A **reflection** flips the graph about one of the axes.

When reflecting, the graph is flipped about one of the axes. It is important to apply this transformation *before* any translation.

Reflection in the x-axis

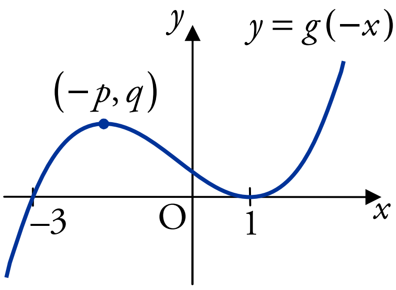
$-f(x)$ reflects the graph of $f(x)$ in the x-axis.

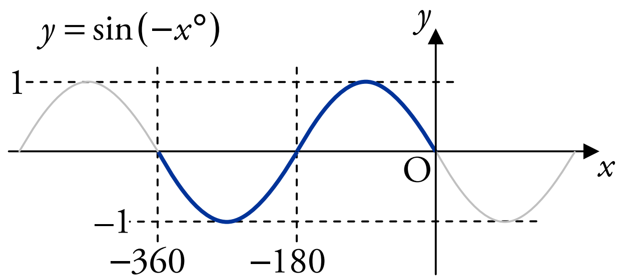




Reflection in the y-axis

$f(-x)$ reflects the graph of $f(x)$ in the y-axis.





From the graphs, $\sin(-x)° = -\sin x°$

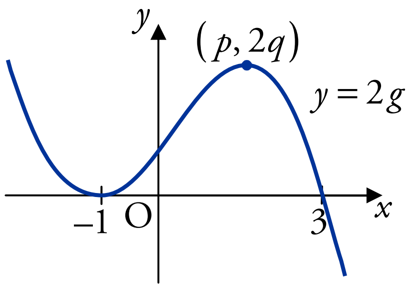
Scaling

A **scaling** stretches or compresses the graph along one of the axes.

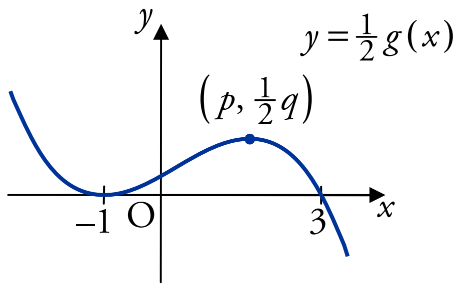
Scaling vertically

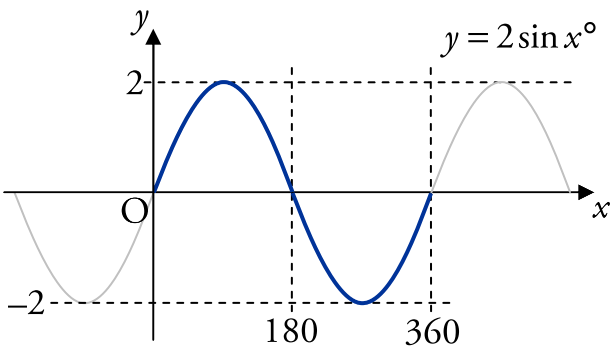
$kf(x)$ scales the graph of $f(x)$ in the vertical direction. The y-coordinate of each point on the graph is multiplied by k, roots are unaffected. These examples consider positive k.

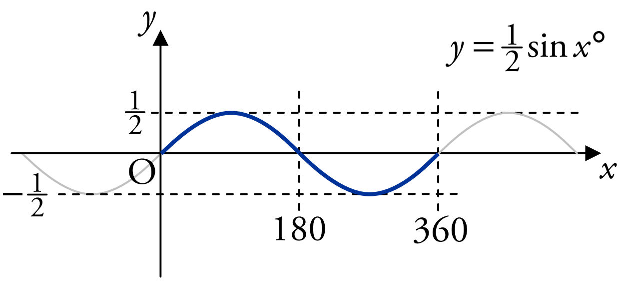
$k > 1$ stretches



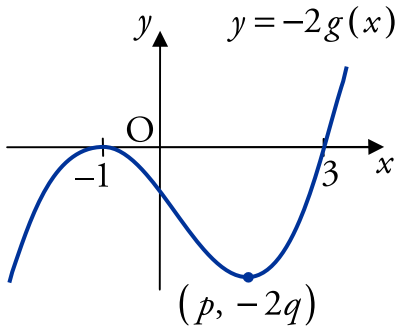
$0 < k < 1$ compresses





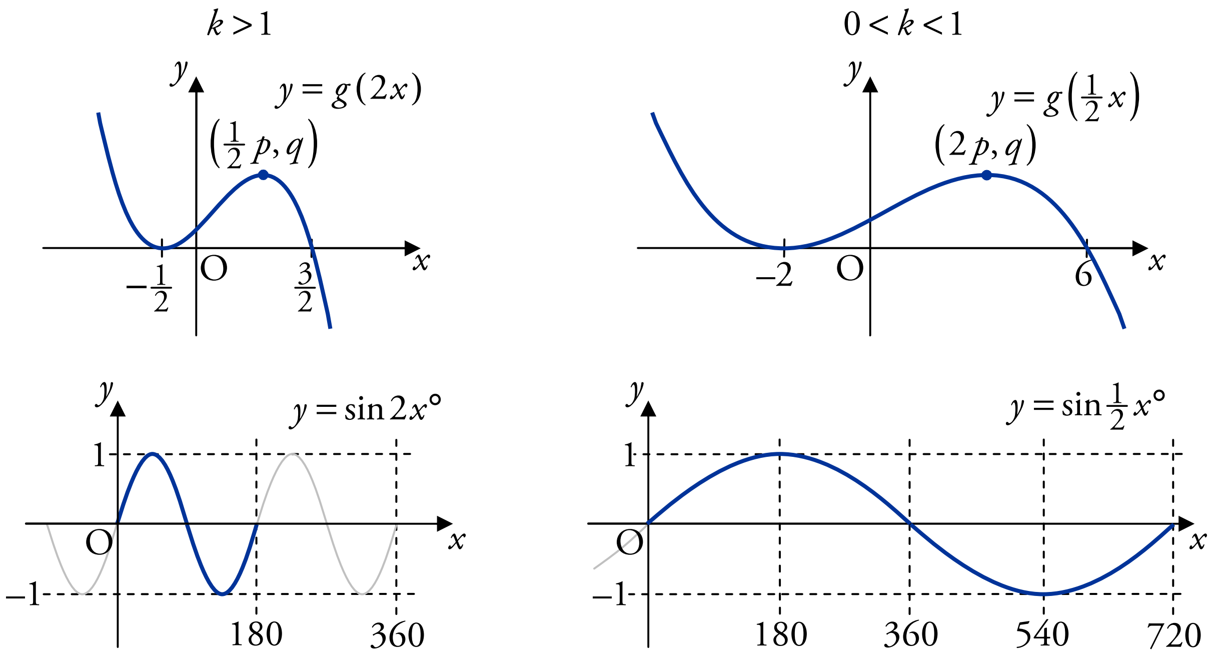


Negative k causes the same scaling, but the graph must then be reflected in the x-axis:

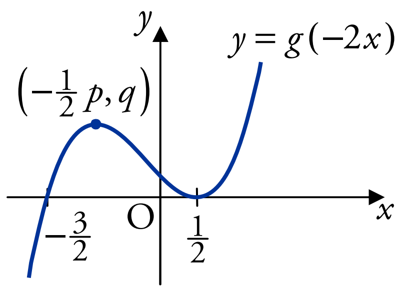


Scaling horizontally

$f(kx)$ scales the graph of $f(x)$ in the horizontal direction. The coordinates of the y-axis intercept stay the same. The examples below consider positive k.

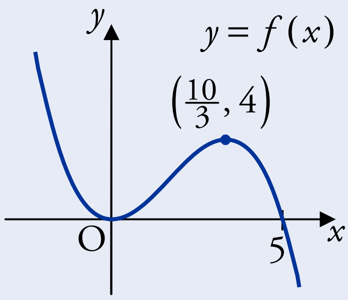


Negative k causes the same scaling, but the graph must then be reflected in the y-axis:



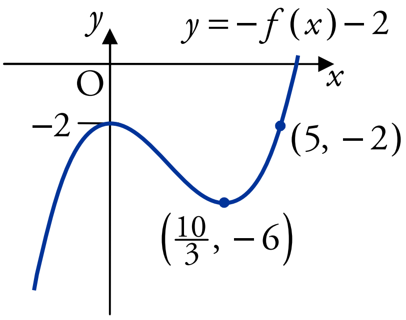
EXAMPLES

1. The graph of $y = f(x)$ is shown below.

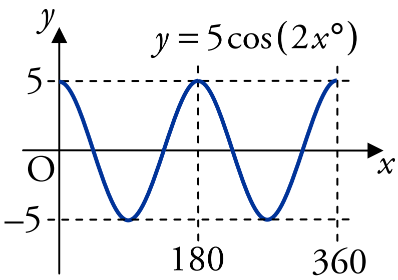


Sketch the graph of $y = -f(x) - 2$.

Reflect in the x-axis, then shift down by 2:



2. Sketch the graph of $y = 5\cos(2x°)$ where $0 \leq x \leq 360$.



Remember

The graph of $y = \cos x$: