



Higher Mathematics

UNIT 1 OUTCOME 1

Straight Lines

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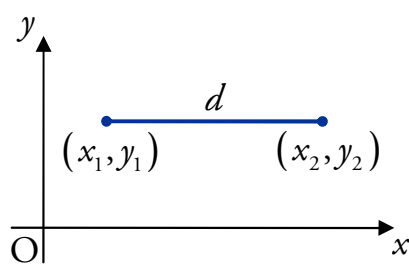
OUTCOME 1

Straight Lines

1 The Distance Between Points

Points on Horizontal or Vertical Lines

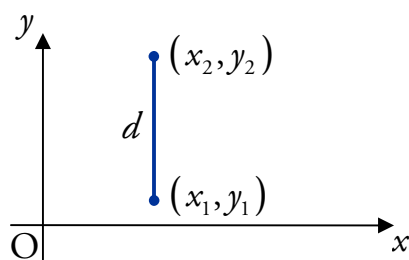
It is relatively straightforward to work out the distance between two points which lie on a line parallel to the x - or y -axis.



In the diagram to the left, the points (x_1, y_1) and (x_2, y_2) lie on a line parallel to the x -axis, i.e. $y_1 = y_2$.

The distance between the points is simply the difference in the x -coordinates, i.e.

$$d = x_2 - x_1 \text{ where } x_2 > x_1.$$



In the diagram to the left, the points (x_1, y_1) and (x_2, y_2) lie on a line parallel to the y -axis, i.e. $x_1 = x_2$.

The distance between the points is simply the difference in the y -coordinates, i.e.

$$d = y_2 - y_1 \text{ where } y_2 > y_1.$$

EXAMPLE

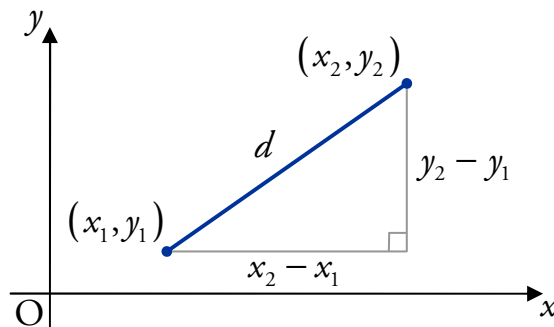
1. Calculate the distance between the points $(-7, -3)$ and $(16, -3)$.

Since both y -coordinates are 3, the distance is the difference in the x -coordinates:

$$\begin{aligned} d &= 16 - (-7) \\ &= 16 + 7 \\ &= 23 \text{ units.} \end{aligned}$$

The Distance Formula

The distance formula gives us a method for working out the length of the straight line between *any* two points. It is based on Pythagoras's Theorem.



Note

The “ $y_2 - y_1$ ” and “ $x_2 - x_1$ ” come from the method above.

The **distance** d between the points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \text{ units.}$$

EXAMPLES

2. A is the point $(-2, 4)$ and B $(3, 1)$. Calculate the length of the line AB.

$$\begin{aligned} \text{The length is } & \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ & = \sqrt{(3 - (-2))^2 + (1 - 4)^2} \\ & = \sqrt{5^2 + (-3)^2} \\ & = \sqrt{25 + 9} \\ & = \sqrt{34} \text{ units.} \end{aligned}$$

3. Calculate the distance between the points $(\frac{1}{2}, -\frac{15}{4})$ and $(-1, -1)$.

$$\begin{aligned} \text{The distance is } & \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ & = \sqrt{(-1 - \frac{1}{2})^2 + (-1 + \frac{15}{4})^2} \\ & = \sqrt{(-\frac{2}{2} - \frac{1}{2})^2 + (-\frac{4}{4} + \frac{15}{4})^2} \\ & = \sqrt{(-\frac{3}{2})^2 + (\frac{11}{4})^2} \\ & = \sqrt{\frac{9}{4} + \frac{121}{16}} \\ & = \sqrt{\frac{36}{16} + \frac{121}{16}} \\ & = \sqrt{\frac{157}{16}} \\ & = \frac{\sqrt{157}}{4} \text{ units.} \end{aligned}$$

Note

You need to become confident working with fractions and surds – so practise!

2 The Midpoint Formula

The point half-way between two points is called their midpoint. It is calculated as follows.

The **midpoint** of (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

It may be helpful to think of the midpoint as the “average” of two points.

EXAMPLES

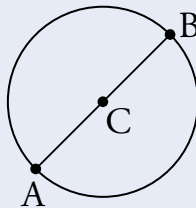
1. Calculate the midpoint of the points $(1, -4)$ and $(7, 8)$.

$$\begin{aligned} \text{The midpoint is } & \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ & = \left(\frac{7 + 1}{2}, \frac{8 + (-4)}{2}\right) \\ & = (4, 2). \end{aligned}$$

Note

Simply writing “The midpoint is $(4, 2)$ ” would be acceptable in an exam.

2. In the diagram below, $A(9, -2)$ lies on the circumference of the circle with centre $C(17, 12)$, and the line AB is a diameter of the circle. Find the coordinates of B .



Since C is the centre of the circle and AB is a diameter, C is the midpoint of AB . Using the midpoint formula, we have:

$$(17, 12) = \left(\frac{9 + x}{2}, \frac{-2 + y}{2}\right) \quad \text{where } B \text{ is the point } (x, y).$$

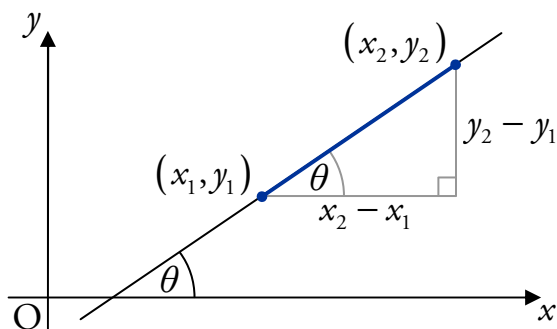
By comparing x - and y -coordinates, we have:

$$\begin{aligned} \frac{9 + x}{2} = 17 & \quad \text{and} \quad \frac{-2 + y}{2} = 12 \\ 9 + x = 34 & \quad -2 + y = 24 \\ x = 25 & \quad y = 26. \end{aligned}$$

So B is the point $(25, 26)$.

3 Gradients

Consider a straight line passing through the points (x_1, y_1) and (x_2, y_2) :



Note

“ θ ” is the Greek letter “theta”. It is often used to stand for an angle.

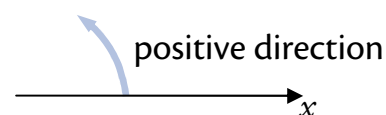
The **gradient** m of the line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\text{change in vertical height}}{\text{change in horizontal distance}} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{for } x_1 \neq x_2.$$

Also, since $\tan \theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{y_2 - y_1}{x_2 - x_1}$ we obtain:

$$m = \tan \theta$$

where θ is the angle between the line and the positive direction of the x -axis.



Note

As a result of the above definitions:

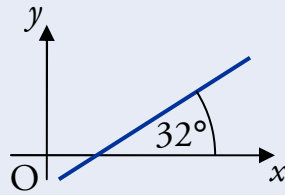
- lines with positive gradients slope up, from left to right;
- lines with negative gradients slope down, from left to right;
- lines parallel to the x -axis have a gradient of zero;
- lines parallel to the y -axis have an undefined gradient.

We may also use the fact that:

Two distinct lines are said to be **parallel** when they have the same gradient (or when both lines are vertical).

EXAMPLES

1. Calculate the gradient of the straight line shown in the diagram below.



$$\begin{aligned} m &= \tan \theta \\ &= \tan 32^\circ \\ &= 0.62 \text{ (to 2 d.p.)} \end{aligned}$$

2. Find the angle that the line joining $P(-2, -2)$ and $Q(1, 7)$ makes with the positive direction of the x -axis.

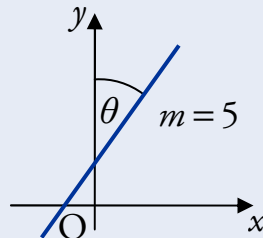
$$\text{The line has gradient } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 + 2}{1 + 2} = 3.$$

$$\text{And so } m = \tan \theta$$

$$\tan \theta = 3$$

$$\theta = \tan^{-1}(3) = 71.57^\circ \text{ (to 2 d.p.)}.$$

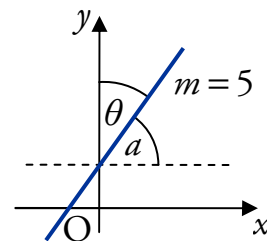
3. Find the size of angle θ shown in the diagram below.



We need to be careful because the θ in the question is *not* the θ in “ $m = \tan \theta$ ”.

So we work out the angle a and use this to find θ :

$$\begin{aligned} a &= \tan^{-1}(m) \\ &= \tan^{-1}(5) \\ &= 78.690^\circ. \end{aligned}$$



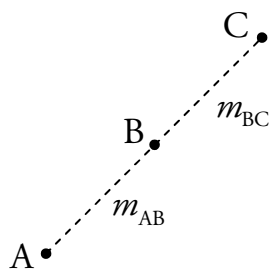
$$\text{So } \theta = 90^\circ - 78.690^\circ = 11.31^\circ \text{ (to 2 d.p.)}.$$

4 Collinearity

Points which lie on the same straight line are said to be **collinear**.

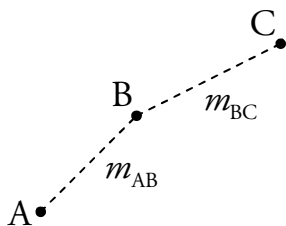
To test if three points A, B and C are collinear we can:

1. Work out m_{AB} .
2. Work out m_{BC} (or m_{AC}).
3. If the gradients from 1. and 2. are the same then A, B and C are collinear.



$m_{AB} = m_{BC}$ so A, B and C are collinear.

If the gradients are different then the points are not collinear.



$m_{AB} \neq m_{BC}$ so A, B and C are not collinear.

This test for collinearity can only be used in two dimensions.

EXAMPLES

1. Show that the points P(-6, -1), Q(0, 2) and R(8, 6) are collinear.

$$m_{PQ} = \frac{2 - (-1)}{0 - (-6)} = \frac{3}{6} = \frac{1}{2} \qquad m_{QR} = \frac{6 - 2}{8 - 0} = \frac{4}{8} = \frac{1}{2}$$

Since $m_{PQ} = m_{QR}$ and Q is a common point, P, Q and R are collinear.

2. The points A(1, -1), B(-1, k) and C(5, 7) are collinear.

Find the value of k.

Since the points are collinear $m_{AB} = m_{AC}$:

$$\frac{k - (-1)}{-1 - 1} = \frac{7 - (-1)}{5 - 1}$$

$$\frac{k + 1}{-2} = \frac{8}{4}$$

$$k + 1 = 2 \times (-2)$$

$$k = -5.$$

5 Gradients of Perpendicular Lines

Two lines at right-angles to each other are said to be **perpendicular**.

If perpendicular lines have gradients m and m_{\perp} then

$$m \times m_{\perp} = -1.$$

Conversely, if $m \times m_{\perp} = -1$ then the lines are perpendicular.

The simple rule is: if you know the gradient of one of the lines, then the gradient of the other is calculated by inverting the gradient (i.e. “flipping” the fraction) and changing the sign. For example:

$$\text{if } m = \frac{2}{3} \quad \text{then } m_{\perp} = -\frac{3}{2}.$$

Note that this rule *cannot* be used if the line is parallel to the x - or y -axis.

- If a line is parallel to the x -axis ($m = 0$), then the perpendicular line is parallel to the y -axis – it has an undefined gradient.
- If a line is parallel to the y -axis then the perpendicular line is parallel to the x -axis – it has a gradient of zero.

EXAMPLES

1. Given that T is the point $(1, -2)$ and S is $(-4, 5)$, find the gradient of a line perpendicular to ST.

$$m_{ST} = \frac{5 - (-2)}{-4 - 1} = -\frac{7}{5}$$

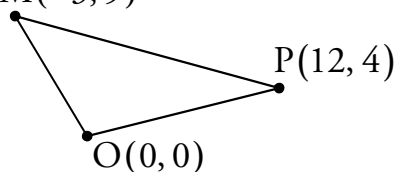
$$\text{So } m_{\perp} = \frac{5}{7} \text{ since } m_{ST} \times m_{\perp} = -1.$$

2. Triangle MOP has vertices $M(-3, 9)$, $O(0, 0)$ and $P(12, 4)$.

Show that the triangle is right-angled.

Sketch:

$M(-3, 9)$



$$\begin{aligned} m_{OM} &= \frac{9-0}{-3-0} & m_{MP} &= \frac{9-4}{-3-12} & m_{OP} &= \frac{4-0}{12-0} \\ &= -3 & &= -\frac{5}{15} & &= \frac{1}{3} \\ & & &= -\frac{1}{3} & & \end{aligned}$$

Since $m_{OM} \times m_{OP} = -1$, OM is perpendicular to OP which means $\triangle MOP$ is right-angled at O.

Note

The converse of Pythagoras's Theorem could also be used here:

$$d_{OP}^2 = 12^2 + 4^2 = 160$$

$$d_{OM}^2 = (-3)^2 + 9^2 = 90$$

$$d_{MP}^2 = (12 - (-3))^2 + (4 - 9)^2$$

$$= 15^2 + (-5)^2$$

$$= 250.$$

Since $d_{OP}^2 + d_{OM}^2 = d_{MP}^2$, triangle MOP is right-angled at O.

6 The Equation of a Straight Line

To work out the equation of a straight line, we need to know two things: the gradient of the line, and a point which lies on the line.

The straight line through the point (a, b) with gradient m has the equation

$$y - b = m(x - a).$$

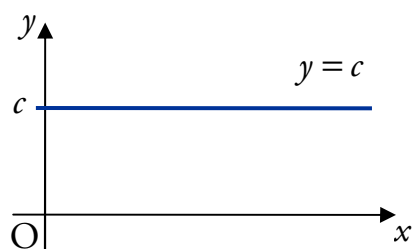
Notice that if we have a point $(0, c)$ – the y -axis intercept – then the equation becomes $y = mx + c$. You should already be familiar with this form.

It is good practice to rearrange the equation of a straight line into the form

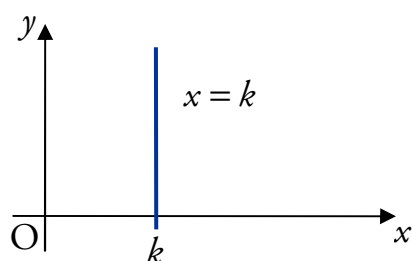
$$ax + by + c = 0$$

where a is positive. This is known as the general form of the equation of a straight line.

Lines Parallel to Axes



If a line is parallel to the x -axis (i.e. $m = 0$), its equation is $y = c$.



If a line is parallel to the y -axis (i.e. m is undefined), its equation is $x = k$.

EXAMPLES

1. Find the equation of the line with gradient $\frac{1}{3}$ passing through the point $(3, -4)$.

$$y - b = m(x - a)$$

$$y - (-4) = \frac{1}{3}(x - 3) \quad (\times 3)$$

$$3y + 12 = x - 3$$

$$3y = x - 15$$

$$x - 3y - 15 = 0.$$

Note

It is usually easier to multiply out the fraction before expanding the brackets.

2. Find the equation of the line passing through $A(3, 2)$ and $B(-2, 1)$.

To work out the equation, we must first find the gradient of the line AB:

$$\begin{aligned} m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - 1}{3 - (-2)} = \frac{1}{5}. \end{aligned}$$

Now we have a gradient, and can use this with one of the given points:

$$y - b = m(x - a)$$

$$y - 2 = \frac{1}{5}(x - 3) \quad \text{using } A(3, 2) \text{ and } m_{AB} = \frac{1}{5}$$

$$5y - 10 = x - 3$$

$$5y = x + 7$$

$$x - 5y + 7 = 0.$$

3. Find the equation of the line passing through $(-\frac{3}{5}, 4)$ and $(-\frac{3}{5}, 5)$.

The gradient is undefined since the x -coordinates are equal.

So the equation of the line is $x = -\frac{3}{5}$.

Extracting the Gradient

You should already be familiar with the following fact.

The line with equation $y = mx + c$ has gradient m .

It is important to remember that you must rearrange the equation of a straight line into this form *before* extracting the gradient.

EXAMPLES

4. Find the gradient of the line with equation $3x + 2y + 4 = 0$.

We have to rearrange the equation:

$$3x + 2y + 4 = 0$$

$$2y = -3x - 4$$

$$y = -\frac{3}{2}x - 2.$$

So the gradient is $-\frac{3}{2}$.

5. The line through points A(3, -3) and B has equation $5x - y - 18 = 0$.

Find the equation of the line through A which is perpendicular to AB.

First, find the gradient of AB:

$$5x - y - 18 = 0$$

$$y = 5x - 18.$$

So $m_{AB} = 5$ and $m_{\perp} = -\frac{1}{5}$. Therefore the equation is:

$$y + 3 = -\frac{1}{5}(x - 3) \quad \text{using A(3, -3) and } m_{\perp} = -\frac{1}{5}$$

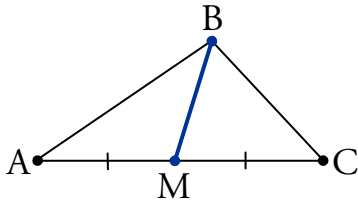
$$5y + 15 = -(x - 3)$$

$$5y + 15 = -x + 3$$

$$x + 5y + 12 = 0.$$

7 Medians

A **median** of a triangle is a line through a vertex and the midpoint of the opposite side.



BM is a median of $\triangle ABC$.

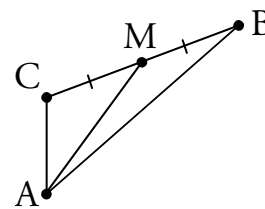
The standard process for finding the equation of a median is shown below.

EXAMPLE

Triangle ABC has vertices $A(4, -9)$, $B(10, 2)$ and $C(4, -4)$.

Find the equation of the median from A.

Start with a sketch:



Step 1

Calculate the midpoint of the relevant line.

Using $B(10, 2)$ and $C(4, -4)$:

$$\begin{aligned} M &= \left(\frac{10+4}{2}, \frac{2+(-4)}{2} \right) \\ &= \left(\frac{14}{2}, \frac{-2}{2} \right) \\ &= (7, -1). \end{aligned}$$

Step 2

Calculate the gradient of the line between the midpoint and the opposite vertex.

Using $A(4, -9)$ and $M(7, -1)$:

$$\begin{aligned} m_{AM} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1 - (-9)}{7 - 4} = \frac{8}{3}. \end{aligned}$$

Step 3

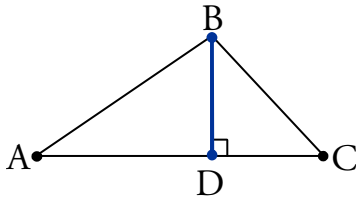
Find the equation using this gradient and either of the two points used in Step 2.

Using $A(4, -9)$ and $m_{AM} = \frac{8}{3}$:

$$\begin{aligned} y - b &= m(x - a) \\ y + 9 &= \frac{8}{3}(x - 4) \quad (\times 3) \\ 3y + 27 &= 8x - 32 \\ 3y &= 8x - 59 \\ 8x - 3y - 59 &= 0. \end{aligned}$$

8 Altitudes

An **altitude** of a triangle is a line through a vertex, perpendicular to the opposite side.



BD is an altitude of $\triangle ABC$.

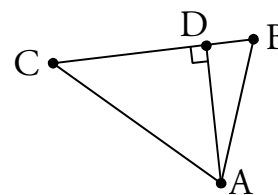
The standard process for finding the equation of an altitude is shown below.

EXAMPLE

Triangle ABC has vertices $A(3, -5)$, $B(4, 3)$ and $C(-7, 2)$.

Find the equation of the altitude from A.

Start with a sketch:



Step 1

Calculate the gradient of the side which is perpendicular to the altitude.

Using $B(4, 3)$ and $C(-7, 2)$:

$$\begin{aligned} m_{BC} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - 3}{-7 - 4} = \frac{1}{11}. \end{aligned}$$

Step 2

Calculate the gradient of the altitude using $m \times m_{\perp} = -1$.

Using $m_{BC} \times m_{AD} = -1$:

$$m_{AD} = -11.$$

Step 3

Find the equation using this gradient and the point that the altitude passes through.

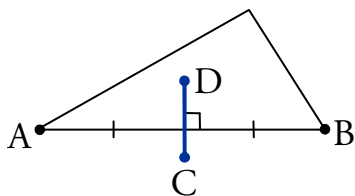
Using $A(3, -5)$ and $m_{AD} = -11$:

$$\begin{aligned} y - b &= m(x - a) \\ y + 5 &= -11(x - 3) \\ y &= -11x + 28 \end{aligned}$$

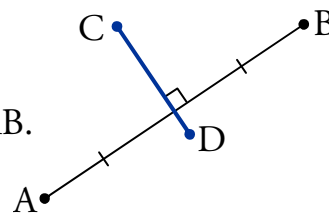
$$11x + y - 28 = 0.$$

9 Perpendicular Bisectors

A **perpendicular bisector** is a line which cuts through the midpoint of a line segment at right-angles.



In both cases, CD is the perpendicular bisector of AB.



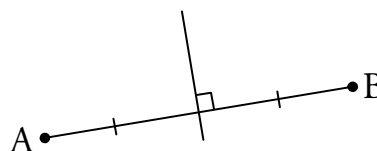
The standard process for finding the equation of a perpendicular bisector is shown below.

EXAMPLE

A is the point $(-2, 1)$ and B is the point $(4, 7)$.

Find the equation of the perpendicular bisector of AB.

Start with a sketch:



Step 1

Calculate the midpoint of the line segment being bisected.

Using $A(-2, 1)$ and $B(4, 7)$:

$$\begin{aligned} \text{Midpoint}_{AB} &= \left(\frac{-2+4}{2}, \frac{1+7}{2} \right) \\ &= (1, 4). \end{aligned}$$

Step 2

Calculate the gradient of the line used in Step 1, then find the gradient of its perpendicular bisector using $m \times m_{\perp} = -1$.

Using $A(-2, 1)$ and $B(4, 7)$:

$$\begin{aligned} m_{AB} &= \frac{7-1}{4-(-2)} \\ &= \frac{6}{6} \\ &= 1. \end{aligned}$$

$$m_{\perp} = -1 \text{ since } m_{AB} \times m_{\perp} = -1.$$

Step 3

Find the equation of the perpendicular bisector using the point from Step 1 and the gradient from Step 2.

Using $(1, 4)$ and $m_{\perp} = -1$:

$$\begin{aligned} y - b &= m(x - a) \\ y - 4 &= -(x - 1) \\ y &= -x + 1 + 4 \\ y &= -x + 5 \end{aligned}$$

$$x + y - 5 = 0.$$

10 Intersection of Lines

Many problems involve lines which intersect (cross each other). Once we have equations for the lines, the problem is to find values for x and y which satisfy both equations, i.e. solve simultaneous equations.

There are three different techniques and, depending on the form of the equations, one may be more efficient than the others.

We will demonstrate these techniques by finding the point of intersection of the lines with equations $3y = x + 15$ and $y = x - 3$.

Elimination

This should be a familiar method, and can be used in all cases.

$$3y = x + 15 \quad \textcircled{1}$$

$$y = x - 3 \quad \textcircled{2}$$

$$\textcircled{1} - \textcircled{2}: 2y = 18$$

$$y = 9.$$

Put $y = 9$ into $\textcircled{2}$: $x = 9 + 3$

$$= 12.$$

So the lines intersect at the point $(12, 9)$.

Equating

This method can be used when both equations have a common x - or y -coefficient. In this case, both equations have an x -coefficient of 1.

Make x the subject of both equations:

$$x = 3y - 15 \qquad x = y + 3.$$

Equate:

$$3y - 15 = y + 3$$

$$2y = 18$$

$$y = 9.$$

Substitute $y = 9$ into:

$$y = x - 3$$

$$x = 9 + 3$$

$$= 12.$$

So the lines intersect at the point $(12, 9)$.

Substitution

This method can be used when one equation has an x - or y -coefficient of 1 (i.e. just an x or y with no multiplier).

Substitute $y = x - 3$ into: Substitute $x = 12$ into:

$$3y = x + 15$$

$$y = x - 3$$

$$3(x - 3) = x + 15$$

$$y = 12 - 3$$

$$3x - 9 = x + 15$$

$$= 9.$$

$$2x = 24$$

$$x = 12.$$

So the lines intersect at the point (12, 9).

EXAMPLE

- Find the point of intersection of the lines $2x - y + 11 = 0$ and $x + 2y - 7 = 0$.

Eliminate y :

$$2x - y + 11 = 0 \quad \text{①}$$

$$x + 2y - 7 = 0 \quad \text{②}$$

$$2 \times \text{①} + \text{②}: 5x + 15 = 0$$

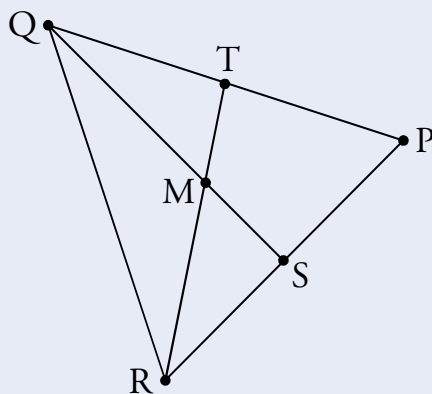
$$x = -3.$$

Put $x = -3$ into ①: $-6 - y + 11 = 0$

$$y = 5.$$

So the point of intersection is $(-3, 5)$.

- Triangle PQR has vertices $P(8, 3)$, $Q(-1, 6)$ and $R(2, -3)$.



- Find the equation of altitude QS.
- Find the equation of median RT.
- Hence find the coordinates of M.

(a) Find the gradient of PR:

$$m_{PR} = \frac{3 - (-3)}{8 - 2} = \frac{6}{6} = 1.$$

So the gradient of QS is $m_{QS} = -1$, since the $m_{PR} \times m_{QS} = -1$.

Find the equation of QS using Q(-1,6) and $m_{QS} = -1$:

$$y - 6 = -1(x + 1)$$

$$y - 6 = -x - 1$$

$$x + y - 5 = 0.$$

(b) Find the coordinates of T, the midpoint of PQ:

$$T = \left(\frac{8 - 1}{2}, \frac{3 + 6}{2} \right) = \left(\frac{7}{2}, \frac{9}{2} \right).$$

Find the gradient of RT using R(2,-3) and $T\left(\frac{7}{2}, \frac{9}{2}\right)$:

$$m_{RT} = \frac{\frac{9}{2} - (-3)}{\frac{7}{2} - 2} = \frac{\frac{15}{2}}{\frac{3}{2}} = \frac{15}{3} = 5.$$

Find the equation of RT using R(2,-3) and $m_{RT} = 5$:

$$y + 3 = 5(x - 2)$$

$$y + 3 = 5x - 10$$

$$5x - y - 13 = 0.$$

(c) Now solve the equations simultaneously to find M.

Eliminate y :

$$x + y - 5 = 0 \quad \text{①}$$

$$5x - y - 13 = 0 \quad \text{②}$$

$$\text{①} + \text{②}: 6x - 18 = 0$$

$$x = 3.$$

Put $x = 3$ into ①: $3 + y - 5 = 0$

$$y = 2.$$

So the point of intersection is M(3,2).

Note

This is the standard method for finding the equation of an altitude.

Note

This is the standard method for finding the equation of a median.

Note

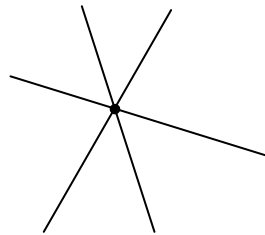
Any of the three techniques could have been used here.

11 Concurrency

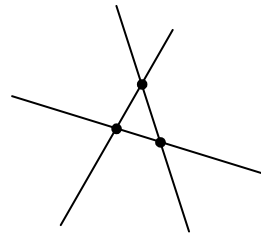
Any number of lines are said to be **concurrent** if there is a point through which they all pass.

So in the previous section, by finding a point of intersection of two lines we showed that the two lines were concurrent.

For *three* lines to be concurrent, they must all pass through a single point.

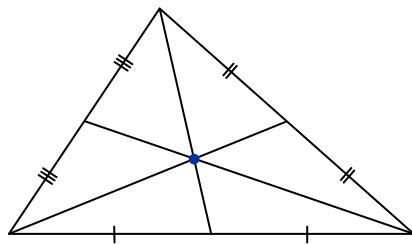


The three lines are concurrent

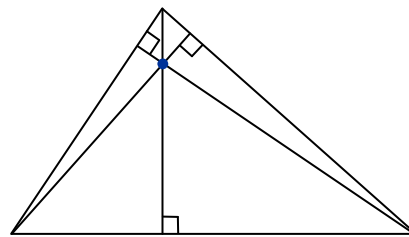


The three lines are *not* concurrent

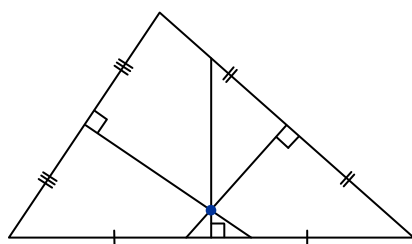
A surprising fact is that the following lines in a triangle are concurrent.



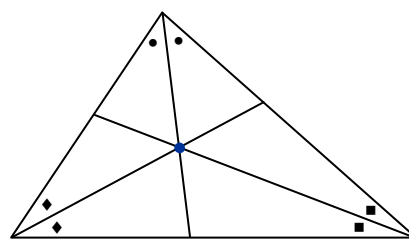
The three medians of a triangle are concurrent.



The three altitudes of a triangle are concurrent.



The three perpendicular bisectors in a triangle are concurrent.



The three angle bisectors of a triangle are concurrent.